`barvinok`: User Guide

Version: barvinok-0.27-41-g882c57e

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1 Internal Representation of the `barvinok` library

Our `barvinok` library is built on top of `PolyLib` (Wilde 1993; Loechner 1999). In particular, it reuses the implementations of the algorithm of Loechner and Wilde (1997) for computing parametric vertices and the algorithm of Clauss and Loechner (1998) for computing chamber decompositions. Initially, our library was meant to be a replacement for the algorithm of Clauss and Loechner (1998), also implemented in `PolyLib`, for computing quasi-polynomials. To ease the transition of application programs we tried to reuse the existing data structures as much as possible.

1.1 Existing Data Structures

Inside `PolyLib` integer values are represented by the `Value` data type. Depending on a configure option, the data type may either by a 32-bit integer, a 64-bit integer or an arbitrary precision integer using `GMP`. The `barvinok` library requires that `PolyLib` is compiled with support for arbitrary precision integers.

The basic structure for representing (unions of) polyhedra is a `Polyhedron`.

```
typedef struct polyhedron {
  unsigned Dimension, NbConstraints, NbRays, NbEq, NbBid;
  Value **Constraint;
  Value **Ray;
  Value *p_Init;
  int p_Init_size;
  struct polyhedron *next;
} Polyhedron;
```

The attribute `Dimension` is the dimension of the ambient space, i.e., the number of variables. The attributes `Constraint` and `Ray` point to two-dimensional arrays of constraints and generators, respectively. The number of rows is stored in `NbConstraints` and `NbRays`, respectively. The number of columns in both arrays is equal to `1+Dimension+1`. The first column of `Constraint` is either 0 or 1 depending on whether the constraint is an equality (0) or an inequality (1). The number of equalities is stored in `NbEq`. If the constraint is $\langle \mathbf{a}, \mathbf{x} \rangle + c \geq 0$, then the next columns contain the coefficients a_i and the final column contains the constant c. The first column of `Ray` is either 0 or 1 depending on whether the generator is a line (0) or a vertex or ray (1). The number of lines is stored in `NbBid`. Let d be the least common multiple (lcm) of the denominators of the coordinates of a vertex $\mathbf{v}$, then the next columns contain dv_i and the final column contains d. For a ray, the final column contains 0. The field `next` points to the next polyhedron in the union of polyhedra. It is `0` if this is the last (or only) polyhedron in the union. For more information on this structure, we refer to Wilde (1993).

Quasi-polynomials are represented using the `evalue` and `enode` structures.

```
typedef enum { polynomial, periodic, evector } enode_type;

typedef struct _evalue {
  Value d;              /* denominator */
  union {
    Value n;            /* numerator (if denominator != 0) */
    struct _enode *p;   /* pointer   (if denominator == 0) */
  } x;
} evalue;

typedef struct _enode {
  enode_type type;      /* polynomial or periodic or evector */
  int size;             /* number of attached pointers */
  int pos;              /* parameter position */
  evalue arr[1];        /* array of rational/pointer */
} enode;
```

If the field `d` of an `evalue` is zero, then the `evalue` is a placeholder for a pointer to an `enode`, stored in `x.p`. Otherwise, the `evalue` is a rational number with numerator `x.n` and denominator `d`. An `enode` is either a `polynomial` or a `periodic`, depending on the value of `type`. The length of the array `arr` is stored in `size`. For a `polynomial`, `arr` contains the coefficients. For a `periodic`, it contains the values for the different residue classes modulo the parameter indicated by `pos`. For a polynomial, `pos` refers to the variable of the polynomial. The value of `pos` is 1 for the first parameter. That is, if the value of `pos` is 1 and the first parameter is p, and if the length of the array is l, then in case it is a polynomial, the `enode` represents

$$\texttt{arr[0]} + \texttt{arr[1]}\,p + \texttt{arr[2]}\,p^2 + \cdots + \texttt{arr[l-1]}\,p^{l-1}.$$

If it is a periodic, then it represents

$$[\texttt{arr[0]}, \texttt{arr[1]}, \texttt{arr[2]}, \ldots, \texttt{arr[l-1]}]_p\,.$$

Note that the elements of a `periodic` may themselves be other `periodics` or even `polynomials`. In our library, we only allow the elements of a `periodic` to be other `periodics` or rational numbers. The chambers and their corresponding quasi-polynomial are stored in `Enumeration` structures.

```
typedef struct _enumeration {
  Polyhedron *ValidityDomain; /* constraints on the parameters */
  evalue EP;                  /* dimension = combined space    */
  struct _enumeration *next;  /* Ehrhart Polynomial,
                                 corresponding to parameter
                                 values inside the domain
                                 ValidityDomain above          */
} Enumeration;
```

For more information on these structures, we refer to Loechner (1999).

Example 1.1 *Figure 1.2 is a skillful reconstruction of Figure 2 from Loechner (1999). It shows the contents of the* `enode` *structures representing the quasi-polynomial* $[1, 2]_p p^2 + 3p + \frac{5}{2}$.

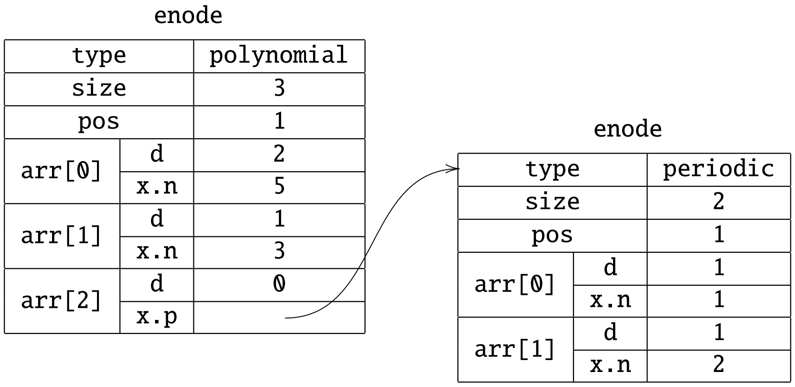


Figure 1.2: The quasi-polynomial $[1,2]_p p^2 + 3p + \frac{5}{2}$.

1.2 Options

The `barvinok_options` structure contains various options that influence the behavior of the library.

```
struct barvinok_options {
    struct barvinok_stats   *stats;

    /* PolyLib options */
    unsigned    MaxRays;

    /* NTL options */
                /* LLL reduction parameter delta=LLL_a/LLL_b */
    long        LLL_a;
    long        LLL_b;

    /* barvinok options */
    #define BV_SPECIALIZATION_BF 2
    #define BV_SPECIALIZATION_DF 1
    #define BV_SPECIALIZATION_RANDOM 0
    #define BV_SPECIALIZATION_TODD 3
    int         incremental_specialization;

    unsigned long           max_index;
    int                     primal;
    int                     lookup_table;
    int                     count_sample_infinite;

    int                     try_Delaunay_triangulation;

    #define     BV_APPROX_SIGN_NONE     0
```

```
#define     BV_APPROX_SIGN_APPROX   1
#define     BV_APPROX_SIGN_LOWER    2
#define     BV_APPROX_SIGN_UPPER    3
int                     polynomial_approximation;
#define     BV_APPROX_NONE          0
#define     BV_APPROX_DROP          1
#define     BV_APPROX_SCALE         2
#define     BV_APPROX_VOLUME        3
#define BV_APPROX_BERNOULLI 4
int                     approximation_method;
#define     BV_APPROX_SCALE_FAST    (1 << 0)
#define     BV_APPROX_SCALE_NARROW  (1 << 1)
#define     BV_APPROX_SCALE_NARROW2 (1 << 2)
#define     BV_APPROX_SCALE_CHAMBER (1 << 3)
int                     scale_flags;
#define     BV_VOL_LIFT             0
#define     BV_VOL_VERTEX           1
#define     BV_VOL_BARYCENTER       2
int                     volume_triangulate;

/* basis reduction options */
#define     BV_GBR_NONE     0
#define     BV_GBR_GLPK     1
#define     BV_GBR_CDD      2
int         gbr_lp_solver;

/* bernstein options */
#define     BV_BERNSTEIN_NONE   0
#define     BV_BERNSTEIN_MAX    1
#define     BV_BERNSTEIN_MIN   -1
int         bernstein_optimize;

#define     BV_BERNSTEIN_FACTORS    1
#define     BV_BERNSTEIN_INTERVALS  2
int         bernstein_recurse;

#define     BV_LP_POLYLIB           0
#define     BV_LP_GLPK              1
#define     BV_LP_CDD               2
#define     BV_LP_CDDF              3
#define     BV_LP_PIP               4
int         lp_solver;

#define     BV_HULL_GBR             0
#define     BV_HULL_HILBERT         1
int         integer_hull;
```

```
};

struct barvinok_options *barvinok_options_new_with_defaults();
```

The function `barvinok_options_new_with_defaults` can be used to create a `barvinok_options` structure with default values.

- `PolyLib` options
  - `MaxRays`

    The value of `MaxRays` is passed to various `PolyLib` functions and defines the maximum size of a table used in the double description computation in the `PolyLib` function `Chernikova`. In earlier versions of `PolyLib`, this parameter had to be conservatively set to a high number to ensure successful operation, resulting in significant memory overhead. Our change to allow this table to grow dynamically is available in recent versions of `PolyLib`. In these versions, the value no longer indicates the maximal table size, but rather the size of the initial allocation. This value may be set to `0` or left as set by `barvinok_options_new_with_defaults`.
- `NTL` options
  - `LLL_a`
  - `LLL_b`

    The values used for the reduction parameter in the call to `NTL`'s implementation of Lenstra, Lenstra and Lovasz' basis reduction algorithm (LLL).
- `barvinok` specific options
  - `incremental_specialization`

    Selects the specialization algorithm to be used. If set to `0` then a direct specialization is performed using a random vector. Value `1` selects a depth first incremental specialization, while value `2` selects a breadth first incremental specialization. The default is selected by the `--enable-incremental` `configure` option. For more information we refer to Verdoolaege (2005, Section 4.4.3).

1.3 Data Structures for Quasi-polynomials

Internally, we do not represent our quasi-polynomials as step-polynomials, but, similarly to Loechner (1999), as polynomials with periodic numbers for coefficients. However, we also allow our periodic numbers to be represented by fractional parts of degree-1 polynomials rather than an explicit enumeration using the `periodic` type. By default, the current version of `barvinok` uses `periodic`s, but this can be changed through the `--enable-fractional` configure option. In the latter case, the quasi-polynomial using fractional parts can also be converted to an actual step-polynomial using `evalue_frac2floor`, but this is not fully supported yet.

For reasons of compatibility,[1] we shoehorned our representations for piecewise quasi-polynomials into the existing data structures. To this effect, we introduced four new types, `fractional`, `relation`, `partition` and `flooring`.

```
typedef enum { polynomial, periodic, evector, fractional,
               relation, partition, flooring } enode_type;
```

The field `pos` is not used in most of these additional types and is therefore set to `-1`.

The types `fractional` and `flooring` represent polynomial expressions in a fractional part or a floor respectively. The generator is stored in `arr[0]`, while the coefficients are stored in the remaining array elements. That is, an `enode` of type `fractional` represents

$$\texttt{arr[1]} + \texttt{arr[2]}\{\texttt{arr[0]}\} + \texttt{arr[3]}\{\texttt{arr[0]}\}^2 + \cdots + \texttt{arr[l-1]}\{\texttt{arr[0]}\}^{l-2}.$$

An `enode` of type `flooring` represents

$$\texttt{arr[1]} + \texttt{arr[2]}\lfloor\texttt{arr[0]}\rfloor + \texttt{arr[3]}\lfloor\texttt{arr[0]}\rfloor^2 + \cdots + \texttt{arr[l-1]}\lfloor\texttt{arr[0]}\rfloor^{l-2}.$$

Example 1.3 *The internal representation of the quasi-polynomial*

$$\left(1 + 2\left\{\frac{p}{2}\right\}\right)p^2 + 3p + \frac{5}{2}$$

is shown in Figure 1.4.

The `relation` type is used to represent strides. In particular, if the value of `size` is 2, then the value of a `relation` is (in pseudo-code):

```
(value(arr[0]) == 0) ? value(arr[1]) : 0
```

If the size is 3, then the value is:

```
(value(arr[0]) == 0) ? value(arr[1]) : value(arr[2])
```

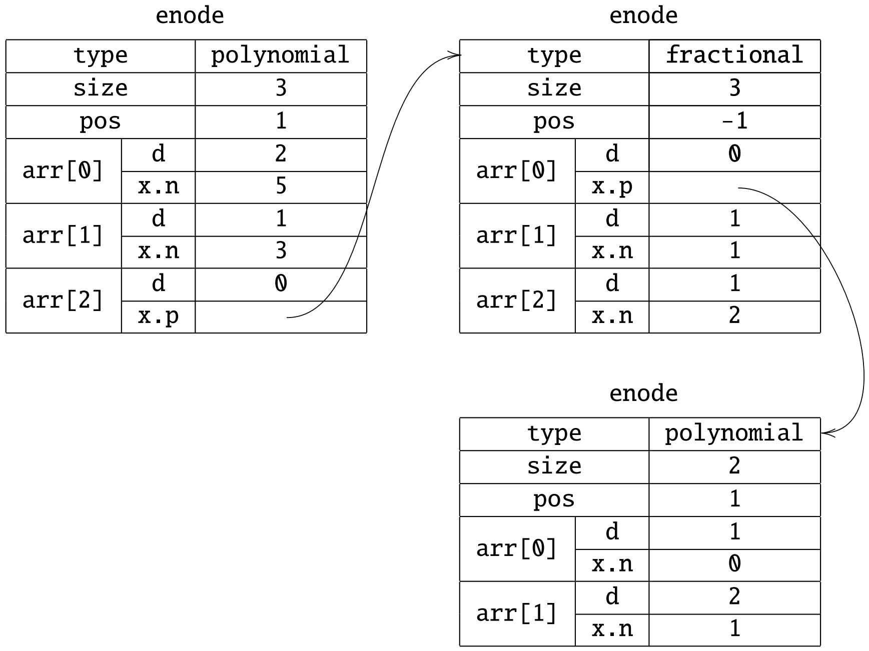
The type of `arr[0]` is typically `fractional`.

Finally, the `partition` type is used to represent piecewise quasi-polynomials. We prefer to encode this information inside `evalue`s themselves rather than using `Enumeration`s since we want to perform the same kinds of operations on both quasi-polynomials and piecewise quasi-polynomials. An `enode` of type `partition` may not be nested inside another `enode`. The size of the array is twice the number of "chambers". Pointers to chambers are stored in the even slots, whereas pointer to the associated quasi-polynomials are stored in the odd slots. To be able to store pointers to chambers, the definition of `evalue` was changed as follows.

```
typedef struct _evalue {
  Value d;              /* denominator */
  union {
    Value n;            /* numerator (if denominator > 0) */
```

---

[1] Also known as laziness.



enode

type		polynomial
size		3
pos		1
arr[0]	d	2
	x.n	5
arr[1]	d	1
	x.n	3
arr[2]	d	0
	x.p	

enode

type		fractional
size		3
pos		-1
arr[0]	d	0
	x.p	
arr[1]	d	1
	x.n	1
arr[2]	d	1
	x.n	2

enode

type		polynomial
size		2
pos		1
arr[0]	d	1
	x.n	0
arr[1]	d	2
	x.n	1

Figure 1.4: The quasi-polynomial $\left(1 + 2\left\{\frac{p}{2}\right\}\right) p^2 + 3p + \frac{5}{2}$.

```
    struct _enode *p;   /* pointer   (if denominator == 0) */
    Polyhedron *D;      /* domain    (if denominator == -1) */
  } x;
} evalue;
```

Note that we allow a “chamber” to be a union of polyhedra as discussed in Verdoolaege (2005, Section 4.5.1). Chambers with extra variables, i.e., those of Verdoolaege (2005, Section 4.6.5), are only partially supported. The field `pos` is set to the actual dimension, i.e., the number of parameters.

1.4 Operations on Quasi-polynomials

In this section we discuss some of the more important operations on `evalue`s provided by the `barvinok` library. Some of these operations are extensions of the functions from `PolyLib` with the same name.

```
void eadd(const evalue *e1,evalue *res);
void emul(const evalue *e1, evalue *res);
```

The functions `eadd` and `emul` takes two (pointers to) `evalue`s `e1` and `res` and computes their sum and product respectively. The result is stored in `res`, overwriting (and deallocating) the original value of `res`. It is an error if exactly one of the arguments of `eadd` is of type `partition` (unless the other argument is `0`). The addition and multiplication operations are described in Verdoolaege (2005, Section 4.5.1) and Verdoolaege (2005, Section 4.5.2) respectively.

The function `eadd` is an extension of the function `new_eadd` from Seghir (2002). Apart from supporting the additional types from Section 1.3, the new version also additionally imposes an order on the nesting of different `enode`s. Without such an ordering, `evalue`s could be constructed representing for example

$$(0y^0 + (0x^0 + 1x^1)y^1)x^0 + (0y^0 - 1y^1)x^1,$$

which is just a funny way of saying 0.

```
void eor(evalue *e1, evalue *res);
```

The function `eor` implements the union operation from Verdoolaege (2005, Section 4.5.3). Both arguments are assumed to correspond to indicator functions.

```
evalue *esum(evalue *E, int nvar);
evalue *evalue_sum(evalue *E, int nvar, unsigned MaxRays);
```

The function `esum` has been superseded by `evalue_sum`. The function `evalue_sum` performs the summation operation from Verdoolaege (2005, Section 4.5.4). The piecewise step-polynomial represented by `E` is summated over its first `nvar` variables. Note that `E` must be zero or of type `partition`. The function returns the result in a newly allocated `evalue`. Note also that `E` needs to have been converted from `fractional`s to `flooring`s using the function `evalue_frac2floor`.

```
void evalue_frac2floor(evalue *e);
```

This function also ensures that the arguments of the `flooring`s are positive in the relevant chambers. It currently assumes that the argument of each `fractional` in the original `evalue` has a minimum in the corresponding chamber.

```
double compute_evalue(const evalue *e, Value *list_args);
Value *compute_poly(Enumeration *en,Value *list_args);
evalue *evalue_eval(const evalue *e, Value *values);
```

The functions `compute_evalue`, `compute_poly` and `evalue_eval` evaluate a (piecewise) quasi-polynomial at a certain point. The argument `list_args` points to an array of `Value`s that is assumed to be long enough. The `double` return value of `compute_evalue` is inherited from `PolyLib`.

```
void print_evalue(FILE *DST, const evalue *e, char **pname);
```

The function `print_evalue` dumps a human-readable representation to the stream pointed to by `DST`. The argument `pname` points to an array of character strings representing the parameter names. The array is assumed to be long enough.

```
int eequal(const evalue *e1, const evalue *e2);
```

The function `eequal` return true (`1`) if its two arguments are structurally identical. I.e., it does *not* check whether the two (piecewise) quasi-polynomial represent the same function.

```
void reduce_evalue (evalue *e);
```

The function `reduce_evalue` performs some simplifications on `evalue`s. Here, we only describe the simplifications that are directly related to the internal representation. Some other simplifications are explained in Verdoolaege (2005, Section 4.7.2). If the highest order coefficients of a `polynomial`, `fractional` or `flooring` are zero (possibly after some other simplifications), then the size of the array is reduced. If only the constant term remains, i.e., the size is reduced to 1 for `polynomial` or to 2 for the other types, then the whole node is replaced by the constant term. Additionally, if the argument of a `fractional` has been reduced to a constant, then the whole node is replaced by its partial evaluation. A `relation` is similarly reduced if its second branch or both its branches are zero. Chambers with zero associated quasi-polynomials are discarded from a `partition`.

1.5 Generating Functions

The representation of rational generating functions uses some basic types from the `NTL` library (Shoup 2004) for representing arbitrary precision integers (`ZZ`) as well as vectors (`vec_ZZ`) and matrices (`mat_ZZ`) of such integers. We further introduces a type `QQ` for representing a rational number and use vectors (`vec_QQ`) of such numbers.

```
struct QQ {
    ZZ n;
    ZZ d;
};

NTL_vector_decl(QQ,vec_QQ);
```

Each term in a rational generating function is represented by a `short_rat` structure.

```
struct short_rat {
    struct {
        /* rows: terms in numerator */
        vec_QQ  coeff;
        mat_ZZ  power;
    } n;
    struct {
        /* rows: factors in denominator */
        mat_ZZ  power;
    } d;
};
```

The fields `n` and `d` represent the numerator and the denominator respectively. Note that in our implementation we combine terms with the same denominator. In the numerator, `short_rat`

`n.coeff`	3	2
	2	1
`n.power`	2	3
	5	-7
`d.power`	1	-3
	0	2

Figure 1.6: Representation of $\left(\frac{3}{2}\, x_0^2 x_1^3 + 2\, x_0^5 x_1^{-7}\right) / \left((1 - x_0 x_1^{-3})(1 - x_1^2)\right)$.

each element of `coeff` and each row of `power` represents a single such term. The vector `coeff` contains the rational coefficients α_i of each term. The columns of `power` correspond to the powers of the variables. In the denominator, each row of `power` corresponds to the power $\mathbf{b}_{ij}$ of a factor in the denominator.

Example 1.5 *Figure 1.6 shows the internal representation of*

$$\frac{\frac{3}{2}\, x_0^2 x_1^3 + 2\, x_0^5 x_1^{-7}}{(1 - x_0 x_1^{-3})(1 - x_1^2)}.$$

The whole rational generating function is represented by a `gen_fun` structure.

```
typedef std::set<short_rat *,
                 short_rat_lex_smaller_denominator > short_rat_list;

struct gen_fun {
    short_rat_list term;
    Polyhedron *context;

    void add(const QQ& c, const vec_ZZ& num, const mat_ZZ& den);
    void add(short_rat *r);
    void add(const QQ& c, const gen_fun *gf,
             barvinok_options *options);
    void substitute(Matrix *CP);
    gen_fun *Hadamard_product(const gen_fun *gf,
                              barvinok_options *options);
    void print(std::ostream& os,
               unsigned int nparam, char **param_name) const;
    operator evalue *() const;
    ZZ coefficient(Value* params, barvinok_options *options) const;
    void coefficient(Value* params, Value* c) const;

    gen_fun(Polyhedron *C);
    gen_fun(Value c);
    gen_fun(const gen_fun *gf);
    ~gen_fun();
};
```

A new `gen_fun` can be constructed either as empty rational generating function (possibly with a given context `C`), as a copy of an existing rational generating function `gf`, or as constant rational generating function with value for the constant term specified by `c`.
The first `gen_fun::add` method adds a new term to the rational generating function, described by the coefficient `c`, the numerator `num` and the denominator `den`. It makes all powers in the denominator lexico-positive, orders them in lexicographical order and inserts the new term in `term` according to the lexicographical order of the combined powers in the denominator. The second `gen_fun::add` method adds `c` times `gf` to the rational generating function.
The method `gen_fun::operator evalue *` performs the conversion from rational generating function to piecewise step-polynomial explained in Verdoolaege (2005, Section 4.5.5). The `Polyhedron context` is the superset of all points where the enumerator is non-zero used during this conversion, i.e., it is the set Q from Verdoolaege (2005, Equation 4.31). If `context` is `NULL` the maximal allowed context is assumed, i.e., the maximal region with lexico-positive rays.
The method `gen_fun::coefficient` computes the coefficient of the term with power given by `params` and stores the result in `c`. This method performs essentially the same computations as `gen_fun::operator evalue *`, except that it adds extra equality constraints based on the specified values for the power.
The method `gen_fun::substitute` performs the monomial substitution specified by the homogeneous matrix `CP` that maps a set of "compressed parameters" (Meister 2004) to the original set of parameters. That is, if we are given a rational generating function $G(\mathbf{z})$ that encodes the explicit function $g(\mathbf{i}')$, where $\mathbf{i}'$ are the coordinates of the transformed space, and `CP` represents the map $\mathbf{i} = A\mathbf{i}' + \mathbf{a}$ back to the original space with coordinates $\mathbf{i}$, then this method transforms the rational generating function to $F(\mathbf{x})$ encoding the same explicit function $f(\mathbf{i})$, i.e.,

$$f(\mathbf{i}) = f(A\mathbf{i}' + \mathbf{a}) = g(\mathbf{i}').$$

This means that the coefficient of the term $\mathbf{x}^{\mathbf{i}} = \mathbf{x}^{A\mathbf{i}'+\mathbf{a}}$ in $F(\mathbf{x})$ should be equal to the coefficient of the term $\mathbf{z}^{\mathbf{i}'}$ in $G(\mathbf{z})$. In other words, if

$$G(\mathbf{z}) = \sum_i \epsilon_i \frac{\mathbf{z}^{\mathbf{v}_i}}{\prod_j (1 - \mathbf{z}^{\mathbf{b}_{ij}})}$$

then

$$F(\mathbf{x}) = \sum_i \epsilon_i \frac{\mathbf{x}^{A\mathbf{v}_i+\mathbf{a}}}{\prod_j (1 - \mathbf{x}^{A\mathbf{b}_{ij}})}.$$

The method `gen_fun::Hadamard_product` computes the Hadamard product of the current rational generating function with the rational generating function `gf`, as explained in Verdoolaege (2005, Section 4.5.2).

1.6 Counting Functions

Our library provides essentially three different counting functions: one for non-parametric polytopes, one for parametric polytopes and one for parametric sets with existential variables. The old versions of these functions have a "`MaxRays`" argument, while the new versions have a more general `barvinok_options` argument. For more information on `barvinok_options`, see Section 1.2.

```
void barvinok_count(Polyhedron *P, Value* result,
                    unsigned NbMaxCons);
void barvinok_count_with_options(Polyhedron *P, Value* result,
                                 struct barvinok_options *options);
```

The function `barvinok_count` or `barvinok_count_with_options` enumerates the non-parametric polytope `P` and returns the result in the `Value` pointed to by `result`, which needs to have been allocated and initialized. If `P` is a union, then only the first set in the union will be taken into account. For the meaning of the argument `NbMaxCons`, see the discussion on `MaxRays` in Section 1.2.

The function `barvinok_enumerate` for enumerating parametric polytopes was meant to be a drop-in replacement of `PolyLib`'s `Polyhedron_Enumerate` function. Unfortunately, the latter has been changed to accept an extra argument in recent versions of `PolyLib` as shown below.

```
Enumeration* barvinok_enumerate(Polyhedron *P, Polyhedron* C,
                                unsigned MaxRays);
extern Enumeration *Polyhedron_Enumerate(Polyhedron *P,
    Polyhedron *C, unsigned MAXRAYS, char **pname);
```

The argument `MaxRays` has the same meaning as the argument `NbMaxCons` above. The argument `P` refers to the $(d + n)$-dimensional polyhedron defining the parametric polytope. The argument `C` is an n-dimensional polyhedron containing extra constraints on the parameter space. Its primary use is to indicate how many of the dimensions in `P` refer to parameters as any constraint in `C` could equally well have been added to `P` itself. Note that the dimensions referring to the parameters should appear *last*. If either `P` or `C` is a union, then only the first set in the union will be taken into account. The result is a newly allocated `Enumeration`. As an alternative we also provide a function (`barvinok_enumerate_ev` or `barvinok_enumerate_with_options`) that returns an `evalue`.

```
evalue* barvinok_enumerate_ev(Polyhedron *P, Polyhedron* C,
                              unsigned MaxRays);
evalue* barvinok_enumerate_with_options(Polyhedron *P,
        Polyhedron* C, struct barvinok_options *options);
```

For enumerating parametric sets with existentially quantified variables, we provide two functions: `barvinok_enumerate_e` and `barvinok_enumerate_pip`.

```
evalue* barvinok_enumerate_e(Polyhedron *P,
          unsigned exist, unsigned nparam, unsigned MaxRays);
```

```
evalue* barvinok_enumerate_e_with_options(Polyhedron *P,
          unsigned exist, unsigned nparam,
          struct barvinok_options *options);
evalue *barvinok_enumerate_pip(Polyhedron *P,
          unsigned exist, unsigned nparam, unsigned MaxRays);
evalue* barvinok_enumerate_pip_with_options(Polyhedron *P,
          unsigned exist, unsigned nparam,
          struct barvinok_options *options);
evalue *barvinok_enumerate_scarf(Polyhedron *P,
          unsigned exist, unsigned nparam,
          struct barvinok_options *options);
```

The first function tries the simplification rules from Verdoolaege (2005, Section 4.6.2) before resorting to the method based on Parametric Integer Programming (PIP) from Verdoolaege (2005, Section 4.6.3). The second function immediately applies the technique from Verdoolaege (2005, Section 4.6.3). The argument `exist` refers to the number of existential variables, whereas the argument `nparam` refers to the number of parameters. The order of the dimensions in `P` is: counted variables first, then existential variables and finally the parameters. The function `barvinok_enumerate_scarf` performs the same computation as the function `barvinok_enumerate_scarf_series` below, but produces an explicit representation instead of a generating function.

```
gen_fun * barvinok_series(Polyhedron *P, Polyhedron* C,
                          unsigned MaxRays);
gen_fun * barvinok_series_with_options(Polyhedron *P,
    Polyhedron* C, barvinok_options *options);
gen_fun *barvinok_enumerate_e_series(Polyhedron *P,
                  unsigned exist, unsigned nparam,
                  barvinok_options *options);
gen_fun *barvinok_enumerate_scarf_series(Polyhedron *P,
                          unsigned exist, unsigned nparam,
                          barvinok_options *options);
```

The function `barvinok_series` or `barvinok_series_with_options` enumerates parametric polytopes in the form of a rational generating function. The polyhedron `P` is assumed to have only revlex-positive rays.
The function `barvinok_enumerate_e_series` computes a generating function for the number of point in the parametric set defined by `P` with `exist` existentially quantified variables using the projection theorem, as explained in subsection 5.24. The function `barvinok_enumerate_scarf_series` computes a generating function for the number of point in the parametric set defined by `P` with `exist` existentially quantified variables, which is assumed to be 2. This function implements the technique of Scarf and Woods (2006) using the neighborhood complex description of Scarf (1981). It is currently restricted to problems with 3 or 4 constraints involving the existentially quantified variables.

1.7 Auxiliary Functions

In this section we briefly mention some auxiliary functions available in the `barvinok` library.

```
void Polyhedron_Polarize(Polyhedron *P);
```

The function `Polyhedron_Polarize` polarizes its argument and is explained in Verdoolaege (2005, Section 4.4.2).

```
int unimodular_complete(Matrix *M, int row);
```

The function `unimodular_complete` extends the first `row` rows of `M` with an integral basis of the orthogonal complement as explained in Section 5.7. Returns non-zero if the resulting matrix is unimodular.

```
int DomainIncludes(Polyhedron *D1, Polyhedron *D2);
```

The function `DomainIncludes` extends the function `PolyhedronIncludes` provided by `PolyLib` to unions of polyhedra. It checks whether every polyhedron in the union `D2` is included in some polyhedron of `D1`.

```
Polyhedron *DomainConstraintSimplify(Polyhedron *P,
                                     unsigned MaxRays);
```

The value returned by `DomainConstraintSimplify` is a pointer to a newly allocated `Polyhedron` that contains the same integer points as its first argument but possibly has simpler constraints. Each constraint $g\langle \mathbf{a}, \mathbf{x} \rangle \geq c$ is replaced by $\langle \mathbf{a}, \mathbf{x} \rangle \geq \left\lceil \frac{c}{g} \right\rceil$, where g is the greatest common divisor (gcd) of the coefficients in the original constraint. The `Polyhedron` pointed to by `P` is destroyed.

```
Polyhedron* Polyhedron_Project(Polyhedron *P, int dim);
```

The function `Polyhedron_Project` projects `P` onto its last `dim` dimensions.

```
Matrix *left_inverse(Matrix *M, Matrix **Eq);
```

The `left_inverse` function computes the left inverse of `M` as explained in Section 5.6.

```
Matrix *Polyhedron_Reduced_Basis(Polyhedron *P,
                                 struct barvinok_options *options);
```

`Polyhedron_Reduced_Basis` computes a generalized reduced basis of `P`, which is assumed to be a polytope, using the algorithm of Cook et al. (1993). See subsection 5.19 for more information. The basis vectors are stored in the rows of the matrix returned.

```
Vector *Polyhedron_Sample(Polyhedron *P,
                          struct barvinok_options *options);
```

`Polyhedron_Sample` returns an integer point of `P` or `NULL` if `P` contains no integer points. The integer point is found using the algorithm of Cook et al. (1993) and uses `Polyhedron_Reduced_Basis` to compute the reduced bases. See subsection 5.19 for more information.

1.8 `bernstein` Data Structures and Functions

The `bernstein` library used `GiNaC` data structures to represent the data it manipulates. In particular, a polynomial is stored in a `GiNaC::ex`, a list of variable or parameter names is stored in a `GiNaC::exvector`, while the parametric vertices or generators are stored in a `GiNaC::matrix`, where the rows refer to the generators and the columns to the coordinates of each generator.

```
namespace bernstein {
GiNaC::exvector constructParameterVector(
    const char * const *param_names, unsigned nbParams);
GiNaC::exvector constructVariableVector(unsigned nbVariables,
                                        const char *prefix);
}
```

The functions `constructParameterVector` and `constructVariableVector` construct a list of variable names either from a list of `char *`s or by suffixing `prefix` with a number starting from 0. Such lists are needed for the functions `domainVertices`, `bernsteinExpansion` and `evalue_bernstein_coefficients`.

```
namespace bernstein {
GiNaC::matrix domainVertices(Param_Polyhedron *PP, Param_Domain *Q,
                             const GiNaC::exvector& params);
}
```

The function `domainVertices` constructs a matrix representing the generators (in this case vertices) of the `Param_Polyhedron` `PP` for the `Param_Domain` `Q`, to be used in a call to `bernsteinExpansion`. The elements of `params` are used in the resulting matrix to refer to the parameters.

```
namespace bernstein {
GiNaC::lst bernsteinExpansion(const GiNaC::matrix& vert,
                              const GiNaC::ex& poly,
                              const GiNaC::exvector& vars,
                              const GiNaC::exvector& params);
}
```

The function `bernsteinExpansion` computes the Bernstein coefficients of the polynomial `poly` over the parametric polytope that is the convex hull of the rows in `vert`. The vectors `vars` and `params` identify the variables (i.e., the coordinates of the space in which the parametric polytope lives) and the parameters, respectively.

```
namespace bernstein {

typedef std::pair< Polyhedron *, GiNaC::lst > guarded_lst;

struct piecewise_lst {
    const GiNaC::exvector vars;
    std::vector<guarded_lst> list;
    /*  0: just collect terms
     *  1: remove obviously smaller terms (maximize)
     * -1: remove obviously bigger terms (minimize)
     */
    int sign;

    piecewise_lst(const GiNaC::exvector& vars);
    piecewise_lst& combine(const piecewise_lst& other);
    void maximize();
    void simplify_domains(Polyhedron *ctx, unsigned MaxRays);
    GiNaC::numeric evaluate(const GiNaC::exvector& values);
    void add(const GiNaC::ex& poly);
}

}
```

A `piecewise_list` structure represents a list of (disjoint) polyhedral domains, each with an associated `GiNaC::lst` of polynomials. The `vars` member contains the variable names of the dimensions of the polyhedral domains.
`piecewise_lst::combine` computes the common refinement of the polyhedral domains in `this` and `other` and associates to each of the resulting subdomains the union of the sets of polynomials associated to the domains from `this` and `other` that contain the subdomain. If the `sign`s of the `piecewise_list`s are not zero, then the (obviously) redundant elements of these sets are removed from the union. The result is stored in `this`.
`piecewise_lst::maximize` removes polynomials from domains that evaluate to a value that is smaller than or equal to the value of some other polynomial associated to the same domain for each point in the domain.
`piecewise_lst::evaluate` "evaluates" the `piecewise_list` by looking for the domain (if any) that contains the point given by `values` and computing the maximal value attained by any of the associated polynomials evaluated at that point.
`piecewise_lst::add` adds the polynomial `poly` to each of the polynomial associated to each of the domains.
`piecewise_lst::simplify_domains` "simplifies" the domains by removing the constraints that are implied by the constraints in `ctx`, basically by calling `PolyLib`'s `DomainSimplify`. Note that you should only do this at the end of your computation. In particular, you do not want to call this method before calling `piecewise_lst::maximize`, since this method will then have less information on the domains to exploit.

```
namespace barvinok {
bernstein::piecewise_lst *evalue_bernstein_coefficients(
    bernstein::piecewise_lst *pl_all, evalue *e,
    Polyhedron *ctx, const GiNaC::exvector& params);
bernstein::piecewise_lst *evalue_bernstein_coefficients(
    bernstein::piecewise_lst *pl_all, evalue *e,
    Polyhedron *ctx, const GiNaC::exvector& params,
    barvinok_options *options);
}
```

The `evalue_bernstein_coefficients` function will compute the Bernstein coefficients of the piecewise parametric polynomial stored in the `evalue e`. The `params` vector specifies the names to be used for the parameters, while the context `Polyhedron ctx` specifies extra constraints on the parameters. The dimension of `ctx` needs to be the same as the length of `params`. The `evalue e` is assumed to be of type `partition` and each of the domains in this `partition` is interpreted as a parametric polytope in the given parameters. The procedure will compute the Bernstein coefficients of the associated polynomial over each such parametric polytope. The resulting `bernstein::piecewise_lst` collects the Bernstein coefficients over all parametric polytopes in `e`. If `pl_all` is not `NULL` then this list will be combined with the list computed by calling `piecewise_lst::combine`. If `bernstein_optimize` is set to `BV_BERNSTEIN_MAX` in `options`, then this combination will remove obviously redundant Bernstein coefficients with respect to upper bound computation and similarly for `BV_BERNSTEIN_MIN`. The default (`BV_BERNSTEIN_NONE`) is to only remove duplicate Bernstein coefficients.

2 Applications included in the `barvinok` distribution

This section describes some application programs provided by the `barvinok` library, available from `http://freshmeat.net/projects/barvinok/`. For compilation instructions we refer to the `README` file included in the distribution.

Common option to all programs:

`--version`	`-V`	print version
`--help`	`-?`	list available options

2.1 `barvinok_count`

The program `barvinok_count` enumerates a non-parametric polytope. It takes one polytope in `PolyLib` notation as input and prints the number of integer points in the polytope. The `PolyLib` notation corresponds to the internal representation of `Polyhedron`s as explained in Section 1.1. The first line of the input contains the number of rows and the number of columns in the `Constraint` matrix. The rest of the input is composed of the elements of the matrix. Recall that the number of columns is two more than the number of variables, where the extra first columns is one or zero depending on whether the constraint is an inequality (≥ 0) or an equality ($= 0$). The next columns contain the coefficients of the variables and the final column contains the constant in the constraint. E.g., the set $S = \{ s \mid s \geq 0 \wedge 2s \leq 13 \}$ from Verdoolaege (2005, Example 38 on page 134) corresponds to the following input and output.

```
> cat S
2 3

1 1 0
1 -2 13
> ./barvinok_count  < S
POLYHEDRON Dimension:1
           Constraints:2  Equations:0  Rays:2  Lines:0
Constraints 2 3
Inequality: [   1   0 ]
Inequality: [  -2  13 ]
Rays 2 3
Vertex: [   0 ]/1
Vertex: [  13 ]/2
   7
```

Note that if you use `PolyLib` version 5.22.0 or newer then the output may look slightly different as the computation of the `Rays` may have been postponed to a later stage. The program `latte2polylib.pl` can be used to convert a polytope from `LattE` (De Loera et al. 2003) notation to `PolyLib` notation.

As an alternative to the constraints based input, the input polytope may also be specified by its `Ray` matrix. The first line of the input contains the single word `vertices`. The second line contains the number of rows and the number of columns in the `Ray` matrix. The rest of the input is composed of the elements of the matrix. Recall that the number of columns is two more than the number of variables, where the extra first columns is one or zero depending on whether the ray is a line or not. The next columns contain the numerators of the coordinates and the final column contains the denominator of the vertex or 0 for a ray. E.g., the above set can also be described as

```
vertices

2 3

1 0 1
1 13 2
```

2.2 barvinok_enumerate

The program `barvinok_enumerate` enumerates a parametric polytope as a piecewise step-polynomial or rational generating function. It takes two polytopes in `PolyLib` notation as input, optionally followed by a list of parameter names. The two polytopes refer to arguments `P` and `C` of the corresponding function. (See Section 1.6.) The following example was taken by Loechner (1999) from Loechner (1997, Chapter II.2).

```
> cat loechner
# Dimension of the matrix:
7 7
# Constraints:
# i j k P Q cte
1 1 0 0 0 0 0 # 0 <= i
1 -1 0 0 1 0 0 # i <= P
1 0 1 0 0 0 0 # 0 <= j
1 1 -1 0 0 0 0 # j <= i
1 0 0 1 0 0 0 # 0 <= k
1 1 -1 -1 0 0 0 # k <= i-j
0 1 1 1 0 -1 0 # Q = i + j + k

# 2 parameters, no constraints.
0 4
> ./barvinok_enumerate < loechner
POLYHEDRON Dimension:5
           Constraints:6  Equations:1  Rays:5  Lines:0
Constraints 6 7
Equality:    [   1   1   1   0  -1   0 ]
Inequality: [   0   1   1   1  -1   0 ]
Inequality: [   0   1   0   0   0   0 ]
Inequality: [   0   0   1   0   0   0 ]
Inequality: [   0  -2  -2   0   1   0 ]
Inequality: [   0   0   0   0   0   1 ]
Rays 5 7
Ray:    [   1   0   1   1   2 ]
```

```
Ray:    [   1   1   0   1   2 ]
Vertex: [   0   0   0   0   0 ]/1
Ray:    [   0   0   0   1   0 ]
Ray:    [   1   0   0   1   1 ]
POLYHEDRON Dimension:2
           Constraints:1  Equations:0  Rays:3  Lines:2
Constraints 1 4
Inequality: [   0   0   1 ]
Rays 3 4
Line:   [   1   0 ]
Line:   [   0   1 ]
Vertex: [   0   0 ]/1
         - P + Q  >= 0
         2P - Q  >= 0
          1 >= 0

( -1/2 * P^2 + ( 1 * Q + 1/2 )
 * P + ( -3/8 * Q^2 + ( -1/2 * {( 1/2 * Q + 0 )
} + 1/4 )
 * Q + ( -5/4 * {( 1/2 * Q + 0 )
} + 1 )
 )
 )
         Q  >= 0
         P - Q  -1 >= 0
          1 >= 0

( 1/8 * Q^2 + ( -1/2 * {( 1/2 * Q + 0 )
} + 3/4 )
 * Q + ( -5/4 * {( 1/2 * Q + 0 )
} + 1 )
 )
```

The output corresponds to

$$\begin{cases} -\frac{1}{2}P^2 + PQ + \frac{1}{2}P - \frac{3}{8}Q^2 + \left(\frac{1}{4} - \frac{1}{2}\left\{\frac{1}{2}Q\right\}\right)Q + 1 - \frac{5}{4}\left\{\frac{1}{2}Q\right\} \\ & \text{if } P \le Q \le 2P \\ \frac{1}{8}Q^2 + \left(\frac{3}{4} - \frac{1}{2}\left\{\frac{1}{2}Q\right\}\right) - \frac{5}{4}\left\{\frac{1}{2}Q\right\} & \text{if } 0 \le Q \le P - 1. \end{cases}$$

The following is an example of Petr Lisoněk.

```
> cat petr
4 6
1 -1 -1 -1 1 0
1 1 -1 0 0 0
1 0 1 -1 0 0
1 0 0 1 0 -1
```

```
0 3
n
> ./barvinok_enumerate --series < petr
POLYHEDRON Dimension:4
           Constraints:5  Equations:0  Rays:5  Lines:0
Constraints 5 6
Inequality: [  -1  -1  -1   1   0 ]
Inequality: [   1  -1   0   0   0 ]
Inequality: [   0   1  -1   0   0 ]
Inequality: [   0   0   1   0  -1 ]
Inequality: [   0   0   0   0   1 ]
Rays 5 6
Ray:    [   1   1   1   3 ]
Ray:    [   1   1   0   2 ]
Ray:    [   1   0   0   1 ]
Ray:    [   0   0   0   1 ]
Vertex: [   1   1   1   3 ]/1
POLYHEDRON Dimension:1
           Constraints:1  Equations:0  Rays:2  Lines:1
Constraints 1 3
Inequality: [   0   1 ]
Rays 2 3
Line:   [   1 ]
Vertex: [   0 ]/1
(n^3)/((1-n) * (1-n) * (1-n^2) * (1-n^3))
```

Options:

`--floor`	`-f`	convert `fractional`s to `flooring`s
`--convert`	`-c`	convert `fractional`s to `periodic`s
`--series`	`-s`	compute rational generating function instead of piecewise step-polynomial
`--explicit`	`-e`	convert computed rational generating function to a piecewise step-polynomial

2.3 `barvinok_enumerate_e`

The program `barvinok_enumerate_e` enumerates a parametric projected set. It takes a single polytope in `PolyLib` notation as input, followed by two lines indicating the number or existential variables and the number of parameters and optionally followed by a list of parameter names. The syntax for the line indicating the number of existential variables is the letter `E` followed by a space and the actual number. For indicating the number of parameters, the letter `P` is used. The following example corresponds to Verdoolaege (2005, Example 36 on page 129).

```
> cat projected
5 6
```

```
#   k   i   j   p   cst
1   0   1   0   0   -1
1   0   -1  0   0   8
1   0   0   1   0   -1
1   0   0   -1  1   0
0   -1  6   9   0   -7

E 2
P 1
> ./barvinok_enumerate_e <projected
POLYHEDRON Dimension:4
           Constraints:5  Equations:1  Rays:4  Lines:0
Constraints 5 6
Equality:    [   1  -6  -9   0   7 ]
Inequality: [   0   1   0   0  -1 ]
Inequality: [   0  -1   0   0   8 ]
Inequality: [   0   0   1   0  -1 ]
Inequality: [   0   0  -1   1   0 ]
Rays 4 6
Vertex: [  50   8   1   1 ]/1
Ray:    [   0   0   0   1 ]
Ray:    [   9   0   1   1 ]
Vertex: [   8   1   1   1 ]/1
exist: 2, nparam: 1
         P  -3 >= 0
          1 >= 0

( 3 * P + 10 )
         P  -1 >= 0
         - P + 2 >= 0

( 8 * P + 0 )
```

Options:

`--floor`	`-f`	convert `fractionals` to `floorings`
`--convert`	`-c`	convert `fractionals` to `periodics`
`--omega`	`-o`	use `Omega` as a preprocessor
`--pip`	`-p`	call `barvinok_enumerate_pip` instead of `barvinok_enumerate_e`

2.4 barvinok_union

The program `barvinok_union` enumerates a union of parametric polytopes. It takes as input the number of parametric polytopes in the union, the polytopes in combined data and parameter space in `PolyLib` notation, the context in parameter space in `PolyLib` notation and optionally a list of parameter names.

Options:

`--series`	`-s`	compute rational generating function instead of piecewise step-polynomial

2.5 `barvinok_ehrhart`

The program `barvinok_ehrhart` computes the Ehrhart quasi-polynomial of a polytope *P*, i.e., a quasi-polynomial in *n* that evaluates to the number of integer points in the dilation of *P* by a factor *n*. The input is the same as that of `barvinok_count`, except that it may be followed by the variable name. The functionality is the same as running `barvinok_enumerate` on the cone over *P* placed at $n = 1$.

Options:

`--floor`	`-f`	convert `fractional`s to `flooring`s
`--convert`	`-c`	convert `fractional`s to `periodic`s
`--series`	`-s`	compute Ehrhart series instead of Ehrhart quasi-polynomial

2.6 `polyhedron_sample`

The program `polyhedron_sample` takes a polytope in `PolyLib` notation and prints an integer point in the polytope if there is one. The point is computed using `Polyhedron_Sample`.

2.7 `polytope_scan`

The program `polytope_scan` takes a polytope in `PolyLib` notation and prints a list of all integer points in the polytope. Unless the `--direct` options is given, the order is based on the reduced basis computed with `Polyhedron_Reduced_Basis`.

Options:

`--direct`	`-d`	list the points in the lexicographical order

2.8 `lexmin`

The program `lexmin` implements an algorithm for performing PIP based on rational generating functions and provides an alternative for the technique of Feautrier (1988), which is based on cutting planes (Gomory 1963). The input is the same as that of the `example` program from `piplib` (Feautrier 2006), except that the value for the "big parameter" needs to be -1, since there is no need for big parameters, and it does not read any options from the input file.

2.9 `barvinok_summate`

Given a piecewise quasi-polynomial, the program `barvinok_summate` computes the sum of the piecewise quasi-polynomial evaluated in all (integer) values of a subset of the variables. The result is an expression in the remaining variables.

The input format corresponds to the *output* format of `barvinok_enumerate` and `barvinok_enumerate_e`. That is, the program expects a list of guarded quasi-polynomials. Each guarded quasi-polynomial consists of a domain and a quasi-polynomial, separated by an empty line. The domain is specified as a list of constraints, each on a separate line, consisting of an affine expression in the variables followed by `>= 0`. Use the `--verbose` option to check that your input was parsed correctly. The list of guarded quasi-polynomials may be preceded by a line specifying the variables over which to sum as `#variables` followed by a comma separated list of variable names.

For example

```
> cat square_p3
#variables x,y
x -2 >= 0
-3x + n + 9 >= 0
y -4 >= 0
-y +5 >= 0

x * y
> ./barvinok_summate < square_p3
         n + 3 >= 0
         - n  -1 >= 0

18         n  >= 0
          1 >= 0

( 1/2 * n^2 + ( -3 * {( 1/3 * n + 0 )
} + 21/2 )
 * n + ( 9/2 * {( 1/3 * n + 0 )
}^2 + -63/2 * {( 1/3 * n + 0 )
} + 45 )
 )
```

Options:

`--variables`		comma separated list of variables over which to sum
`--verbose`	`-v`	print parsed piecewise quasi-polynomial
`--summation`		specifies which summation method to use; `box` refers to the method of Verdoolaege (2005, Section 4.5.4), `bernoulli` refers to the method of subsection 5.13, `euler` refers to the method of subsection 5.14, and `laurent` refers to the method of subsection 5.15.

2.10 `barvinok_bound`

Given a piecewise quasi-polynomial, the program `barvinok_bound` computes an upper bound (or lower bound) for the values attained by the piecewise quasi-polynomial over all (integer) values of a subset of the variables. The result is an expression in the remaining variables.

The input format corresponds to the *output* format of `barvinok_enumerate` and `barvinok_enumerate_e`. That is, the program expects a list of guarded quasi-polynomials. Each guarded quasi-polynomial consists of a domain and a quasi-polynomial, separated by an empty line. The domain is specified as a list of constraints, each on a separate line, consisting of an affine expression in the variables followed by `>= 0`. Use the `--verbose` option to check that your input was parsed correctly. The list of guarded quasi-polynomials may be preceded by a line specifying the variables over which to compute the upper bound as `#variables` followed by a comma separated list of variable names.

```
> cat devos
#variables V
         U + 2V + 3 >= 0
         - U -2V  >= 0
         - U  10 >= 0
         U  >= 0

( {( 1/3 * U + ( 2/3 * V + 0 ) ) } )
> ./barvinok_bound < devos
(1*U >= 0 && -1*U + 10 >= 0) ? ((2.0/3.0)) : 0
```

Options:

`--variables`		comma separated list of variables over which to compute a bound
`--verbose`	`-v`	print parsed piecewise quasi-polynomial
`--lower`		compute lower bound instead of upper bound

2.11 `polytope_minimize`

The program `polytope_minimize` takes a polytope in `PolyLib` notation and a linear objective function as input and prints an integer point in the polytope attaining the minimial value of the objective function. The objective function is specified as the length of the vector (the number of variables) followed by the coefficients of the variables. The point is computed as explained in subsubsection 5.20.2.

For example

```
> cat min_test
8 8
   1   34    0    0    0    1    0    0
   1    0  -82   -1    0    0    0    0
   1    0  -82    0    0    0   -1    0
   1    0   31    0    0    1    0    0
   1    0    0    0    2   -3    0    0
   1    0    0    0    0   -1    0    0
   1    0    0    0    0    0    0    1
   1   -34 4676  34   -34  21  34  34

6
  34 -4676  -34   34  -21  -34
> ./polytope_minimize < min_test
```

```
7
   2    2 -164  -93  -62 -164    1
```

2.12 `polyhedron_integer_hull`

The program `polyhedron_integer_hull` takes a polyhedron in `PolyLib` notation and prints its integer hull. The integer hull is computed as explained in subsection 5.20.

2.13 `polytope_lattice_width`

The program `polytope_lattice_width` computes the lattice width of a parametric polytope. The input is the same as that of `barvinok_enumerate`. The lattice width is computed as explained in subsection 5.22.

Options:

`--direction`	`-d`	print the lattice width directions

3 `polymake` clients

The `barvinok` distribution includes a couple of `polymake` (Gawrilow and Joswig 2000) clients in the `polymake` subdir.

- `lattice_points` <file>

  Computes the property `LATTICE_POINTS` of a polytope, the number of lattice points in the polytope.

- `h_star_vector` <file>

  Computes the property `H_STAR_VECTOR` of a lattice polytope, the h^*-vector of the polytope (Stanley 1993).

4 `Omega` interface

The `barvinok` distribution includes an interface to `Omega` (Kelly et al. 1996b) `occ`, an extension of `oc` (Kelly et al. 1996a). The extension adds the operations shown in Figure 4.1. Here are some examples:

```
symbolic n, m;
P := { [i,j] : 0 <= i <= n and i <= j <= m };
card P;

P := {[i,j] : 0 <= i < 4*n-1 and 0 <= j < n and
              n-1 <= i+j <= 3*n-2 };
C1 := {[i,j] : 0 <= i < 4*n-1 and 0 <= j < n and
               2*n-1 <= i+j <= 4*n-2 and i <= 2*n-1 };

count_lexsmaller P within C1;

vertices C1;

bmax { [i] -> 2*n*i - n*n + 3*n - 1/2*i*i - 3/2*i-1 :
        (exists j : 0 <= i < 4*n-1 and 0 <= j < n and
                    2*n-1 <= i+j <= 4*n-2 and i <= 2*n-1 ) };

sum { [i,j] -> i*j + n*i*i*j : i,j >= 0 and 5i + 27j <= n+m };
```

Name	Syntax	Explanation
Card	`card` *r*	Computes the number of integer points in *r* and prints the result to standard output
Card	`card` *r* `using parker`	Computes the number of integer points in *r* and prints the result to standard output using the method of Parker and Chatterjee (2004)
Ranking	`ranking` *r*	Computes the rank function of *r* and prints the result to standard output (Loechner et al. 2002; Turjan et al. 2002)
Predecessors	`count_lexsmaller` *r* `within` *d*	Computes a function from the elements of *d* to the number of elements of *r* that are lexicographically smaller than that element and prints the result to standard output.
Vertices	`vertices` *r*	Computes the parametric vertices of *r* using `PolyLib` (Loechner 1999).
Bernstein	`bmax` *f*	Computes the Bernstein coefficients of the function *f* over its domain and removes the redundant coefficients by calling `piecewise_lst::maximize`. The results are printed to standard output. See the example for how to specify the function f.
Sum	`sum` *f*	Computes the sum of the given polynomial *f* over its domain using `barvinok_summate`.

Figure 4.1: Extra relational operations of `occ`

5 Implementation details

5.1 An interior point of a polyhedron

We often need a point that lies in the interior of a polyhedron. The function `inner_point` implements the following algorithm. Each polyhedron P can be written as the sum of a polytope P' and a cone C (the recession cone or characteristic cone of P). Adding a positive multiple of the sum of the extremal rays of C to the barycenter

$$\frac{1}{N}\sum_i \mathbf{v}_i(\mathbf{p})$$

of P', where N is the number of vertices, results in a point in the interior of P.

5.2 The integer points in the fundamental parallelepiped of a simple cone

This section is based on Barvinok (1992, Lemma 5.1) and De Loera and Köppe (2006).

In this section we will deal exclusively with simple cones, i.e. d-dimensional cones with d extremal rays and d facets. Some of the facets of these cones may be open. Since we will mostly be dealing with cones in their explicit representation, we will have occasion to speak of "open rays", by which we will mean that the facet not containing the ray is open. (There is only one such facet because the cone is simple.)

Definition 5.1 (Fundamental parallelepiped) *Let* $K = \mathbf{v} + \operatorname{pos}\{\,\mathbf{u}_i\,\}$ *be a closed (shifted) cone, then the* fundamental parallelepiped Π *of* K *is*

$$\Pi = \mathbf{v} + \left\{ \sum_i \alpha_i \mathbf{u}_i \mid 0 \le \alpha_i < 1 \right\}.$$

If some of the rays $\mathbf{u}_i$ *of* K *are open, then the constraints on the corresponding coefficient* α_i *are such that* $0 < \alpha_i \le 1$.

Lemma 5.2 (Integer points in the fundamental parallelepiped of a simple cone) *Let* $K = \mathbf{v} + \operatorname{pos}\{\,\mathbf{u}_i\,\}$ *be a closed simple cone and let* A *be the matrix with the generators* $\mathbf{u}_i$ *of* K *as rows. Furthermore let* $VAW^{-1} = S = \operatorname{diag}\mathbf{s}$ *be the Smith Normal Form (SNF) of* A*. Then the integer points in the fundamental parallelepiped of* K *are given by*

$$\begin{aligned} \mathbf{w}^T &= \mathbf{v}^T + \left\{ (\mathbf{k}^T W - \mathbf{v}^T) A^{-1} \right\} A && (5.3) \\ &= \mathbf{v}^T + \sum_{i=1}^{d} \left\{ \langle \sum_{j=1}^{d} k_j \mathbf{w}_j^T - \mathbf{v}^T, \mathbf{u}_i^* \rangle \right\} \mathbf{u}_i^T, \end{aligned}$$

where $\mathbf{u}_i^*$ *are the columns of* A^{-1} *and* $k_j \in \mathbb{Z}$ *ranges over* $0 \le k_j < s_j$.

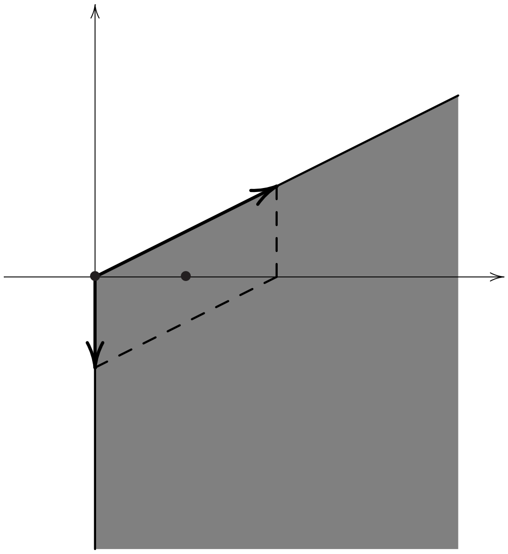


Figure 5.4: The integer points in the fundamental parallelepiped of K

Proof Since $0 \le \{x\} < 1$, it is clear that each such $\mathbf{w}$ lies inside the fundamental parallelepiped. Furthermore,

$$\begin{aligned}
\mathbf{w}^T &= \mathbf{v}^T + \left\{(\mathbf{k}^T W - \mathbf{v}^T)A^{-1}\right\} A \\
&= \mathbf{v}^T + \left((\mathbf{k}^T W - \mathbf{v}^T)A^{-1} - \left\lfloor(\mathbf{k}^T W - \mathbf{v}^T)A^{-1}\right\rfloor\right) A \\
&= \underbrace{\mathbf{k}^T W}_{\in\mathbb{Z}^{1\times d}} - \underbrace{\left\lfloor(\mathbf{k}^T W - \mathbf{v}^T)A^{-1}\right\rfloor}_{\in\mathbb{Z}^{1\times d}} \underbrace{A}_{\in\mathbb{Z}^{d\times d}} \in \mathbb{Z}^{1\times d}.
\end{aligned}$$

Finally, if two such $\mathbf{w}$ are equal, i.e., $\mathbf{w}_1 = \mathbf{w}_2$, then

$$\begin{aligned}
\mathbf{0}^T = \mathbf{w}_1^T - \mathbf{w}_2^T &= \mathbf{k}_1^T W - \mathbf{k}_2^T W + \mathbf{p}^T A \\
&= (\mathbf{k}_1^T - \mathbf{k}_2^T)\, W + \mathbf{p}^T V^{-1} S\, W,
\end{aligned}$$

with $\mathbf{p} \in \mathbb{Z}^d$, or $\mathbf{k}_1 \equiv \mathbf{k}_2 \mod \mathbf{s}$, i.e., $\mathbf{k}_1 = \mathbf{k}_2$. Since $\det S = \det A$, we obtain all points in the fundamental parallelepiped by taking all $\mathbf{k} \in \mathbb{Z}^d$ satisfying $0 \le k_j < s_j$. □

If the cone K is not closed then the coefficients of the open rays should be in $(0, 1]$ rather than in $[0, 1)$. In (5.3), we therefore need to replace the fractional part $\{x\} = x - \lfloor x \rfloor$ by $\{\{x\}\} = x - \lceil x - 1 \rceil$ for the open rays.

Example 5.5 *Let K be the cone*

$$K = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \operatorname{pos}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\},$$

shown in Figure 5.4. Then

$$A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 0 & -1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

We have $\det A = \det S = 2$ *and* $\mathbf{k}_1^T = \begin{bmatrix} 0 & 0 \end{bmatrix}$ *and* $\mathbf{k}_2^T = \begin{bmatrix} 0 & 1 \end{bmatrix}$. *Therefore,*

$$\mathbf{w}_1^T = \left\{ \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 0 & -1 \end{bmatrix} \right\} \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

and

$$\begin{aligned} \mathbf{w}_2^T &= \left\{ \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 0 & -1 \end{bmatrix} \right\} \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}. \end{aligned}$$

5.3 Barvinok's decomposition of simple cones in primal space

As described by De Loera et al. (2004), the first implementation of Barvinok's counting algorithm applied Barvinok's decomposition (Barvinok 1994) in the dual space. Brion's polarization trick (Brion 1988) then ensures that you do not need to worry about lower-dimensional faces in the decomposition. Another way of avoiding the lower-dimensional faces, in the primal space, is to perturb the vertex of the cone such that none of the lower-dimensional face encountered contain any integer points (Köppe 2007). In this section, we describe another technique that is based on allowing some of the facets of the cone to be open.

The basic step in Barvinok's decomposition is to replace a d-dimensional simple cone $K = \operatorname{pos}\{\, \mathbf{u}_i \,\}_{i=1}^d \subset \mathbb{Q}^d$ by a signed sum of (at most) d cones K_j with a smaller determinant (in absolute value). The cones are obtained by successively replacing each generator of K by an appropriately chosen $\mathbf{w} = \sum_{i=1}^d \alpha_i \mathbf{u}_i$, i.e.,

$$K_j = \operatorname{pos}\left(\{\, \mathbf{u}_i \,\}_{i=1}^d \setminus \{\, \mathbf{u}_j \,\} \cup \{\, \mathbf{w} \,\} \right). \tag{5.6}$$

To see that we can use these K_j to perform a decomposition, rearrange the $\mathbf{u}_i$ such that for all $1 \le i \le k$ we have $\alpha_i < 0$ and for all $k+1 \le i \le d'$ we have $\alpha_i > 0$, with $d - d'$ the number of zero α_i. We may assume $k < d'$; otherwise replace $\mathbf{w} \in B$ by $-\mathbf{w} \in B$. We have

$$\mathbf{w} + \sum_{i=1}^{k} (-\alpha_i)\mathbf{u}_i = \sum_{i=k+1}^{d'} \alpha_i \mathbf{u}_i$$

or

$$\sum_{i=0}^{k} \beta_i \mathbf{u}_i = \sum_{i=k+1}^{d'} \alpha_i \mathbf{u}_i, \tag{5.7}$$

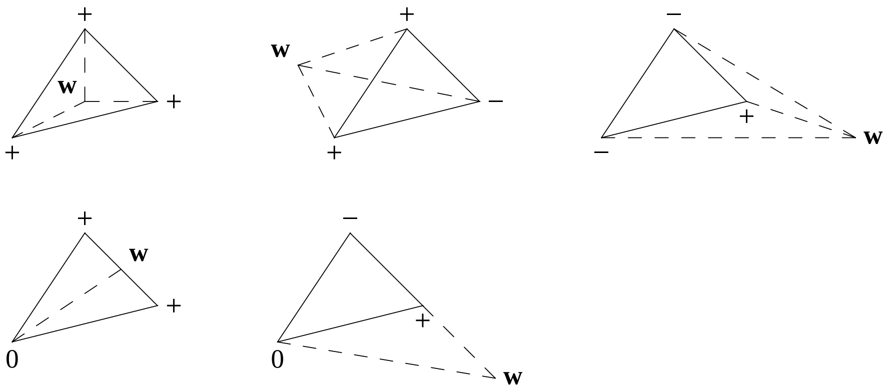


Figure 5.11: Possible locations of **w** with respect to the rays of a 3-dimensional cone. The figure shows a section of the cones.

with $\mathbf{u}_0 = \mathbf{w}$, $\beta_0 = 1$ and $\beta_i = -\alpha_i > 0$ for $1 \le i \le k$. Any two $\mathbf{u}_j$ and $\mathbf{u}_l$ on the same side of the equality are on opposite sides of the linear hull H of the other $\mathbf{u}_i$s since there exists a convex combination of $\mathbf{u}_j$ and $\mathbf{u}_l$ on this hyperplane. In particular, since α_j and α_l have the same sign, we have

$$\frac{\alpha_j}{\alpha_j + \alpha_l}\mathbf{u}_j + \frac{\alpha_l}{\alpha_j + \alpha_l}\mathbf{u}_l \in H \qquad \text{for } \alpha_i \alpha_l > 0. \tag{5.8}$$

The corresponding cones K_j and K_l (with $K_0 = K$) therefore intersect in a common face $F \subset H$. Let

$$K' := \operatorname{pos}\Big(\{\, \mathbf{u}_i \,\}_{i=1}^{d} \cup \{\, \mathbf{w} \,\}\Big),$$

then any $\mathbf{x} \in K'$ lies both in some cone K_i with $0 \le i \le k$ and in some cone K_i with $k + 1 \le i \le d'$. (Just subtract an appropriate multiple of Equation (5.7).) The cones $\{\, K_i \,\}_{i=0}^{k}$ and $\{\, K_i \,\}_{i=k+1}^{d'}$ therefore both form a triangulation of K' and hence

$$[K'] = [K] + \sum_{i=1}^{k} [K_i] - \sum_{j \in J_1} \left[F_j\right] = \sum_{i=k+1}^{d'} [K_i] - \sum_{j \in J_2} \left[F_j\right] \tag{5.9}$$

or

$$[K] = \sum_{i=1}^{d'} \varepsilon_i \, [K_i] + \sum_{j} \delta_j \left[F_j\right], \tag{5.10}$$

with $\varepsilon_i = -1$ for $1 \le i \le k$, $\varepsilon_i = 1$ for $k + 1 \le i \le d'$, $\delta_j \in \{-1, 1\}$ and F_j some lower-dimensional faces. Figure 5.11 shows the possible configurations in the case of a 3-dimensional cone.

As explained above there are several ways of avoiding the lower-dimensional faces in (5.10). Here we will apply the following proposition.

Proposition 5.12 (Köppe and Verdoolaege (2008)) *Let*

$$\sum_{i \in I_1} \epsilon_i [P_i] + \sum_{i \in I_2} \delta_k [P_i] = 0 \tag{5.13}$$

be a (finite) linear identity of indicator functions of closed polyhedra $P_i \subseteq \mathbb{Q}^d$*, where the polyhedra* P_i *with* $i \in I_1$ *are full-dimensional and those with* $i \in I_2$ *lower-dimensional. Let each closed polyhedron be given as*

$$P_i = \left\{ \mathbf{x} \mid \langle \mathbf{b}^*_{i,j}, \mathbf{x} \rangle \geq \beta_{i,j} \textit{ for } j \in J_i \right\}.$$

Let $\mathbf{y} \in \mathbb{Q}^d$ *be a vector such that* $\langle \mathbf{b}^*_{i,j}, \mathbf{y} \rangle \neq 0$ *for all* $i \in I_1 \cup I_2$, $j \in J_i$*. For each* $i \in I_1$*, we define the half-open polyhedron*

$$\begin{aligned} \tilde{P}_i = \Big\{ \mathbf{x} \in \mathbb{Q}^d \mid{} & \langle \mathbf{b}^*_{i,j}, \mathbf{x} \rangle \geq \beta_{i,j} \textit{ for } j \in J_i \textit{ with } \langle \mathbf{b}^*_{i,j}, \mathbf{y} \rangle > 0, \\ & \langle \mathbf{b}^*_{i,j}, \mathbf{x} \rangle > \beta_{i,j} \textit{ for } j \in J_i \textit{ with } \langle \mathbf{b}^*_{i,j}, \mathbf{y} \rangle < 0 \Big\}. \end{aligned} \tag{5.14}$$

Then

$$\sum_{i \in I_1} \epsilon_i [\tilde{P}_i] = 0. \tag{5.15}$$

When applying this proposition to (5.10), we obtain

$$\left[\tilde{K}\right] = \sum_{i=1}^{d'} \varepsilon_i \left[\tilde{K}_i\right], \tag{5.16}$$

where we start out from a given $\tilde{K}$, which may be K itself, i.e., a fully closed cone, or the result of a previous application of the proposition, either through a triangulation (Section 5.4) or a previous decomposition. In either case, a suitable $\mathbf{y}$ is available, either as an interior point of the cone or as the vector used in the previous application (which may require a slight perturbation if it happens to lie on one of the new facets of the cones K_i). We are, however, free to construct a new $\mathbf{y}$ on each application of the proposition. In fact, we will not even construct such a vector explicitly, but rather apply a set of rules that is equivalent to a valid choice of $\mathbf{y}$. Below, we will present an "intuitive" motivation for these rules. For a more algebraic, shorter, and arguably simpler motivation we refer to Köppe and Verdoolaege (2008).

The vector $\mathbf{y}$ has to satisfy $\langle \mathbf{b}^*_j, \mathbf{y} \rangle > 0$ for normals $\mathbf{b}^*_j$ of closed facets and $\langle \mathbf{b}^*_j, \mathbf{y} \rangle < 0$ for normals $\mathbf{b}^*_j$ of open facets of $\tilde{K}$. These constraints delineate a non-empty open cone R from which $\mathbf{y}$ should be selected. For some of the new facets of the cones $\tilde{K}_j$, the cone R will not be cut by the affine hull of the facet. The closedness of these facets is therefore predetermined by $\tilde{K}$. For the other facets, a choice will have to be made. To be able to make the choice based on local information and without computing an explicit vector $\mathbf{y}$, we use the following convention. We first assign an arbitrary total order to the rays. If (the affine hull of) a facet separates the two rays not on the facet $\mathbf{u}_i$ and $\mathbf{u}_j$, i.e., $\alpha_i \alpha_j > 0$ (5.8), then we choose $\mathbf{y}$ to lie on the side of the smallest ray, according to the chosen order. That is, $\langle \tilde{\mathbf{n}}_{ij}, \mathbf{y} \rangle > 0$, for $\tilde{\mathbf{n}}_{ij}$ the normal of the facet pointing towards this smallest ray. Otherwise, i.e., if $\alpha_i\alpha_j < 0$, the interior of K will lie on one side of the facet and then we choose $\mathbf{y}$ to lie on the other side. That is, $\langle \tilde{\mathbf{n}}_{ij}, \mathbf{y}\rangle > 0$, for $\tilde{\mathbf{n}}_{ij}$ the normal of the facet pointing away from the cone K. Figure 5.17 shows some example decompositions with an explicitly marked $\mathbf{y}$.

To see that there is a $\mathbf{y}$ satisfying the above constraints, we need to show that $R \cap S$ is non-empty, with $S = \{\mathbf{y} \mid \langle \tilde{\mathbf{n}}_{i_k j_k}, \mathbf{y}\rangle > 0 \text{ for all } k\}$. It will be easier to show this set is non-empty when the $\mathbf{u}_i$ form an orthogonal basis. Applying a non-singular linear transformation T does not change the decomposition of $\mathbf{w}$ in terms of the $\mathbf{u}_i$ (i.e., the α_i remain unchanged), nor does this change any of the scalar products in the constraints that define $R \cap S$ (the normals are transformed by $\left(T^{-1}\right)^T$). Finding a vector $\mathbf{y} \in T(R \cap S)$ ensures that $T^{-1}(\mathbf{y}) \in R \cap S$. Without loss of generality, we can therefore assume for the purpose of showing that $R \cap S$ is non-empty that the $\mathbf{u}_i$ indeed form an orthogonal basis.

In the orthogonal basis, we have $\mathbf{b}_i^* = \mathbf{u}_i$ and the corresponding inward normal $\mathbf{N}_i$ is either $\mathbf{u}_i$ or $-\mathbf{u}_i$. Furthermore, each normal of a facet of S of the first type is of the form $\tilde{\mathbf{n}}_{i_k j_k} = a_k \mathbf{u}_{i_k} - b_k \mathbf{u}_{j_k}$, with $a_k, b_k > 0$ and $i_k < j_k$, while for the second type each normal is of the form $\tilde{\mathbf{n}}_{i_k j_k} = -a_k \mathbf{u}_{i_k} - b_k \mathbf{u}_{j_k}$, with $a_k, b_k > 0$. If $\tilde{\mathbf{n}}_{i_k j_k} = a_k \mathbf{u}_{i_k} - b_k \mathbf{u}_{j_k}$ is the normal of a facet of S then either $(\mathbf{N}_{i_k}, \mathbf{N}_{j_k}) = (\mathbf{u}_{i_k}, \mathbf{u}_{j_k})$ or $(\mathbf{N}_{i_k}, \mathbf{N}_{j_k}) = (-\mathbf{u}_{i_k}, -\mathbf{u}_{j_k})$. Otherwise, the facet would not cut R. Similarly, if $\tilde{\mathbf{n}}_{i_k j_k} = -a_k \mathbf{u}_{i_k} - b_k \mathbf{u}_{j_k}$ is the normal of a facet of S then either $(\mathbf{N}_{i_k}, \mathbf{N}_{j_k}) = (\mathbf{u}_{i_k}, -\mathbf{u}_{j_k})$ or $(\mathbf{N}_{i_k}, \mathbf{N}_{j_k}) = (-\mathbf{u}_{i_k}, \mathbf{u}_{j_k})$. Assume now that $R \cap S$ is empty, then there exist $\lambda_k, \mu_i \geq 0$ not all zero such that $\sum_k \lambda_k \tilde{\mathbf{n}}_{i_k j_k} + \sum_l \mu_i \mathbf{N}_i = \mathbf{0}$. Assume $\lambda_k > 0$ for some facet of the first type. If $\mathbf{N}_{j_k} = -\mathbf{u}_{j_k}$, then $-b_k$ can only be canceled by another facet k' of the first type with $j_k = i_{k'}$, but then also $\mathbf{N}_{j_{k'}} = -\mathbf{u}_{j_{k'}}$. Since the j_k are strictly increasing, this sequence has to stop with a strictly positive coefficient for the largest $\mathbf{u}_{j_k}$ in this sequence. If, on the other hand, $\mathbf{N}_{i_k} = \mathbf{u}_{i_k}$, then a_k can only be canceled by the normal of a facet k' of the second kind with $i_k = j_{k'}$, but then $\mathbf{N}_{i_{k'}} = -\mathbf{u}_{i_{k'}}$ and we return to the first case. Finally, if $\lambda_k > 0$ only for normals of facets of the second type, then either $\mathbf{N}_{i_k} = -\mathbf{u}_{i_k}$ or $\mathbf{N}_{j_k} = -\mathbf{u}_{j_k}$ and so the coefficient of one of these basis vectors will be strictly negative. That is, the sum of the normals will never be zero and the set $R \cap S$ is non-empty.

For each ray $\mathbf{u}_j$ of cone K_i, i.e., the cone with $\mathbf{u}_i$ replaced by $\mathbf{w}$, we now need to determine whether the facet not containing this ray is closed or not. We denote the (inward) normal of this cone by $\mathbf{n}_{ij}$. Note that cone K_j (if it appears in (5.9), i.e., $\alpha_j \neq 0$) has the same facet opposite $\mathbf{u}_i$ and its normal $\mathbf{n}_{ji}$ will be equal to either $\mathbf{n}_{ij}$ or $-\mathbf{n}_{ij}$, depending on whether we are dealing with an "external" facet, i.e., a facet of K', or an "internal" facet. If, on the other hand, $\alpha_j = 0$, then $\mathbf{n}_{ij} = \mathbf{n}_{0j}$. If $\langle \mathbf{n}_{ij}, \mathbf{y}\rangle > 0$, then the facet is closed. Otherwise it is open. It follows that the two (or more) occurrences of external facets are either all open or all closed, while for internal facets, exactly one is closed.

First consider the facet not containing $\mathbf{u}_0 = \mathbf{w}$. If $\alpha_i > 0$, then $\mathbf{u}_i$ and $\mathbf{w}$ are on the same side of the facet and so $\mathbf{n}_{i0} = \mathbf{n}_{0i}$. Otherwise, $\mathbf{n}_{i0} = -\mathbf{n}_{i0}$. Second, if $\alpha_j = 0$, then replacing $\mathbf{u}_i$ by $\mathbf{w}$ does not change the affine hull of the facet and so $\mathbf{n}_{ij} = \mathbf{n}_{0j}$. Now consider the case that $\alpha_i\alpha_j < 0$, i.e., $\mathbf{u}_i$ and $\mathbf{u}_j$ are on the same side of the hyperplane through the other rays. If we project $\mathbf{u}_i$, $\mathbf{u}_j$ and $\mathbf{w}$ onto a plane orthogonal to the ridge through the other rays, then the possible locations of $\mathbf{w}$ with respect to $\mathbf{u}_i$ and $\mathbf{u}_j$ are

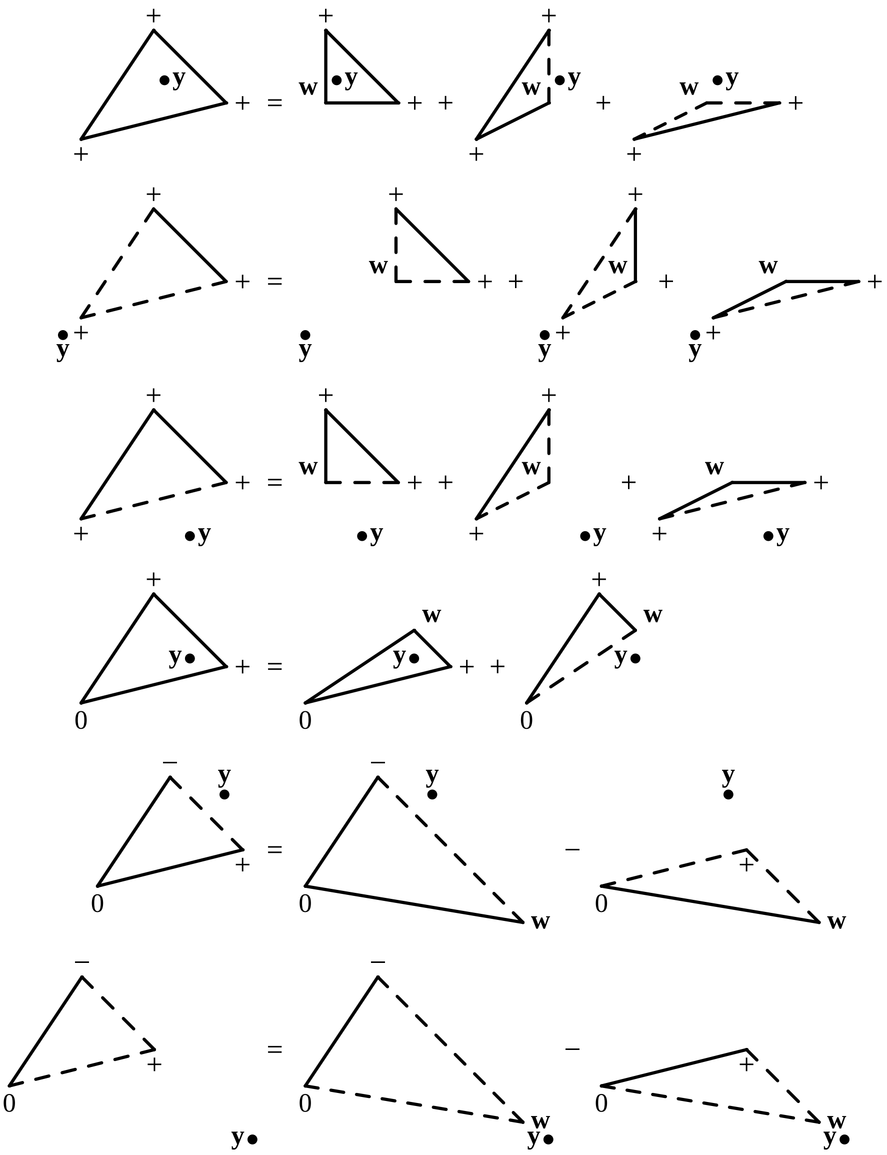


Figure 5.17: Examples of decompositions in primal space.

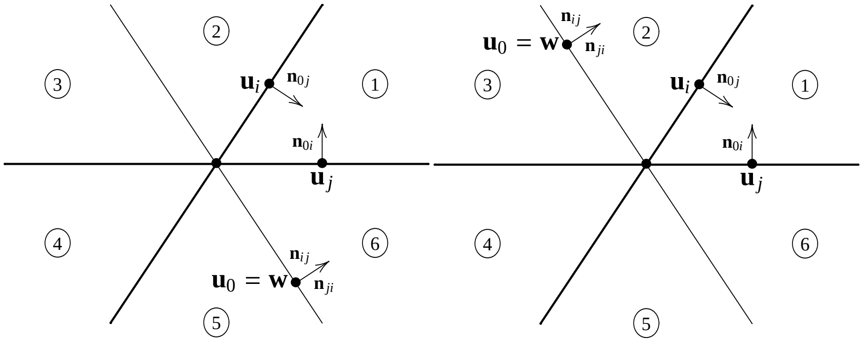


Figure 5.18: Possible locations of $\mathbf{w}$ with respect to $\mathbf{u}_i$ and $\mathbf{u}_j$, projected onto a plane orthogonal to the other rays, when $\alpha_i \alpha_j < 0$.

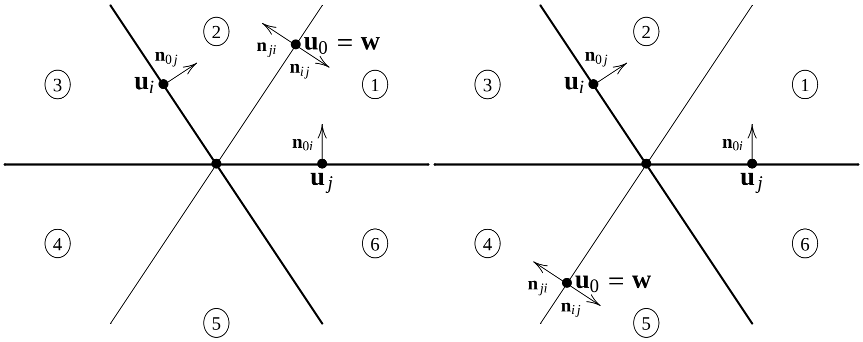


Figure 5.19: Possible locations of $\mathbf{w}$ with respect to $\mathbf{u}_i$ and $\mathbf{u}_j$, projected onto a plane orthogonal to the other rays, when $\alpha_i \alpha_j > 0$.

shown in Figure 5.18. If both $\mathbf{n}_{0i}$ and $\mathbf{n}_{0j}$ are closed then $\mathbf{y}$ lies in region 1 and therefore $\mathbf{n}_{ij}$ (as well as $\mathbf{n}_{ji}$) is closed too. Similarly, if both $\mathbf{n}_{0i}$ and $\mathbf{n}_{0j}$ are open then so is $\mathbf{n}_{ij}$. If only one of the facets is closed, then, as explained above, we choose $\mathbf{n}_{ij}$ to be open, i.e., we take $\mathbf{y}$ to lie in region 3 or 5. Figure 5.19 shows the possible configurations for the case that $\alpha_i \alpha_j > 0$. If exactly one of $\mathbf{n}_{0i}$ and $\mathbf{n}_{0j}$ is closed, then $\mathbf{y}$ lies in region 3 or region 5 and therefore $\mathbf{n}_{ij}$ is closed iff $\mathbf{n}_{0j}$ is closed. Otherwise, as explained above, we choose $\mathbf{n}_{ij}$ to be closed if $i < j$.

The algorithm is summarized in Algorithm 1, where we use the convention that in cone K_i, $\mathbf{u}_i$ refers to $\mathbf{u}_0 = \mathbf{w}$. Note that we do not need any of the rays or normals in this code. The only information we need is the closedness of the facets in the original cone and the signs of the α_i.

5.4 Triangulation in primal space

As in the case for Barvinok's decomposition (Section 5.3), we can transform a triangulation of a (closed) cone into closed simple cones into a triangulation of half-open simple cones that fully partitions the original cone, i.e., such that the half-open sim-

Algorithm 1 Determine whether the facet opposite $\mathbf{u}_j$ is closed in K_i.

```
if $\alpha_j = 0$
    closed[$K_i$][$\mathbf{u}_j$] := closed[$\tilde{K}$][$\mathbf{u}_j$]
else if $i = j$
    if $\alpha_j > 0$
        closed[$K_i$][$\mathbf{u}_j$] := closed[$\tilde{K}$][$\mathbf{u}_j$]
    else
        closed[$K_i$][$\mathbf{u}_j$] := $\neg$closed[$\tilde{K}$][$\mathbf{u}_j$]
else if $\alpha_i \alpha_j > 0$
    if closed[$\tilde{K}$][$\mathbf{u}_i$] = closed[$\tilde{K}$][$\mathbf{u}_j$]
        closed[$K_i$][$\mathbf{u}_j$] := $i < j$
    else
        closed[$K_i$][$\mathbf{u}_j$] := closed[$\tilde{K}$][$\mathbf{u}_j$]
else
    closed[$K_i$][$\mathbf{u}_j$] := closed[$\tilde{K}$][$\mathbf{u}_i$] and closed[$\tilde{K}$][$\mathbf{u}_j$]
```

ple cones do not intersect at their facets. Again, we apply Proposition 5.12 with $\mathbf{y}$ an interior point of the cone (Section 5.1). Note that the interior point $\mathbf{y}$ may still intersect some of the internal facets, so we may need to perturb it slightly. In practice, we apply a lexicographical rule: for such (internal) facets, which always appear in pairs, we close the one with a lexico-positive normal and open the one with a lexico-negative normal.

5.5 Multivariate quasi-polynomials as lists of polynomials

There are many definitions for a (univariate) quasi-polynomial. Ehrhart (1977) uses a definition based on *periodic number*s.

Definition 5.20 *A rational* periodic number $U(p)$ *is a function* $\mathbb{Z} \to \mathbb{Q}$, *such that there exists a* period q *such that* $U(p) = U(p')$ *whenever* $p \equiv p' \mod q$.

Definition 5.21 *A (univariate)* quasi-polynomial f *of degree* d *is a function*

$$f(n) = c_d(n)\, n^d + \cdots + c_1(n)\, n + c_0,$$

where $c_i(n)$ *are rational periodic numbers. I.e., it is a polynomial expression of degree* d *with rational periodic numbers for coefficients. The* period *of a quasi-polynomial is the lcm of the periods of its coefficients.*

Other authors (e.g., Stanley 1986) use the following definition of a quasi-polynomial.

Definition 5.22 *A function* $f : \mathbb{Z} \to \mathbb{Q}$ *is a (univariate)* quasi-polynomial *of period* q *if there exists a list of* q *polynomials* $g_i \in \mathbb{Q}[T]$ *for* $0 \leq i < q$ *such that*

$$f(s) = g_i(s) \qquad \text{if } s \equiv i \mod q.$$

The functions g_i *are called the* constituents.

In our implementation, we use Definition 5.21, but whereas Ehrhart (1977) uses a list of q rational numbers enclosed in square brackets to represent periodic numbers, our periodic numbers are polynomial expressions in fractional parts (Section 1.3). These fractional parts naturally extend to multivariate quasi-polynomials. The bracketed ("explicit") periodic numbers can be extended to multiple variables by nesting them (e.g., Loechner 1999).

Definition 5.22 could be extended in a similar way by having a constituent for each residue modulo a vector period $\mathbf{q}$. However, as pointed out by Woods (2006), this may not result in the minimum number of constituents. A vector period can be considered as a lattice with orthogonal generators and the number of constituents is equal to the index or determinant of that lattice. By considering more general lattices, we can potentially reduce the number of constituents.

Definition 5.23 *A function* $f : \mathbb{Z}^n \to \mathbb{Q}$ *is a (multivariate)* quasi-polynomial *of period* L *if there exists a list of* $\det L$ *polynomials* $g_{\mathbf{i}} \in \mathbb{Q}[T_1, \ldots, T_n]$ *for* $\mathbf{i}$ *in the fundamental parallelepiped of* L *such that*

$$f(\mathbf{s}) = g_{\mathbf{i}}(\mathbf{s}) \qquad \textit{if } \mathbf{s} \equiv \mathbf{i} \mod L.$$

To compute the period lattice from a fractional representation, we compute the appropriate lattice for each fractional part and then take their intersection. Recall that the argument of each fractional part is an affine expression in the parameters $(\langle \mathbf{a}, \mathbf{p} \rangle + c)/m$, with $\mathbf{a} \in \mathbb{Z}^n$ and $c, m \in \mathbb{Z}$. Such a fractional part is translation invariant over any (integer) value of $\mathbf{p}$ such that $\langle \mathbf{a}, \mathbf{p} \rangle + mt = 0$, for some $\mathbf{t} \in \mathbb{Z}$. Solving this homogeneous equation over the integers (in our implementation, we use `PolyLib`'s `SolveDiophantine`) gives the general solution

$$\begin{bmatrix} \mathbf{p} \\ t \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \mathbf{x} \qquad \text{for } \mathbf{x} \in \mathbb{Z}^n.$$

The matrix $U_1 \in \mathbb{Z}^{n \times n}$ then has the generators of the required lattice as columns. The constituents are computed by plugging in each integer point in the fundamental parallelepiped of the lattice. These points themselves are computed as explained in Section 5.2. Note that for computing the constituents, it is sufficient to take any representative of the residue class. For example, we could take $\mathbf{w}^T = \mathbf{k}^T W$ in the notations of Lemma 5.2.

Example 5.24 (Woods (2006)) *Consider the parametric polytope*

$$P_{s,t} = \{ x \mid 0 \le x \le (s+t)/2 \}.$$

The enumerator of $P_{s,t}$ *is*

$$\begin{cases} \frac{s}{2} + \frac{t}{2} + 1 & \textit{if} \begin{bmatrix} s \\ t \end{bmatrix} \in \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix} \mathbb{Z}^2 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \frac{s}{2} + \frac{t}{2} + \frac{1}{2} & \textit{if} \begin{bmatrix} s \\ t \end{bmatrix} \in \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix} \mathbb{Z}^2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \end{cases}$$

The corresponding output of `barvinok_enumerate` *is*

```
         s + t  >= 0
          1 >= 0

Lattice:
[[-1 1]
[-2 0]
]
[0 0]
( 1/2 * s + ( 1/2 * t + 1 )
 )
[-1 0]
( 1/2 * s + ( 1/2 * t + 1/2 )
 )
```

5.6 Left inverse of an affine embedding

We often map a polytope onto a lower dimensional space to remove possible equalities in the polytope. These maps are typically represented by the inverse, mapping the coordinates $\mathbf{x}'$ of the lower-dimensional space to the coordinates $\mathbf{x}$ of (an affine subspace of) the original space, i.e.,

$$\begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} = \begin{bmatrix} T & \mathbf{v} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}' \\ 1 \end{bmatrix},$$

where, as usual in `PolyLib`, we work with homogeneous coordinates. To obtain the transformation that maps the coordinates of the original space to the coordinates of the lower dimensional space, we need to compute the left inverse of the above affine embedding, i.e., an A, $\mathbf{b}$ and d such that

$$d \begin{bmatrix} \mathbf{x}' \\ 1 \end{bmatrix} = \begin{bmatrix} A & \mathbf{b} \\ \mathbf{0}^T & d \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}$$

To compute this left inverse, we first compute the (right) Hermite Normal Form (HNF) of T,

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} T = \begin{bmatrix} H \\ 0 \end{bmatrix}.$$

The left inverse is then simply

$$\begin{bmatrix} dH^{-1}U_1 & -dH^{-1}\mathbf{v} \\ \mathbf{0}^T & d \end{bmatrix}.$$

We often also want a description of the affine subspace that is the range of the affine embedding and this is given by

$$\begin{bmatrix} U_2 & -U_2\mathbf{v} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} = \mathbf{0}.$$

This computation is implemented in `left_inverse`.

5.7 Integral basis of the orthogonal complement of a linear subspace

Let $M_1 \in \mathbb{Z}^{m \times n}$ be a basis of a linear subspace. We first extend M_1 with zero rows to obtain a square matrix M' and then compute the (left) HNF of M',

$$\begin{bmatrix} M_1 \\ 0 \end{bmatrix} = \begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}.$$

The rows of Q_2 span the orthogonal complement of the given subspace. Since Q_2 can be extended to a unimodular matrix, these rows form an integral basis.

If the entries on the diagonal of H are all 1 then M_1 can be extended to a unimodular matrix, by concatenating M_1 and Q_2. The resulting matrix is unimodular, since

$$\begin{bmatrix} M_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} H & 0 \\ 0 & I_{n-m,n-m} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}.$$

This method for extending a matrix of which only a few lines are known to a unimodular matrix is more general than the method described by Bik (1996), which only considers extending a matrix given by a single row.

5.8 Ensuring a polyhedron has only revlex-positive rays

The `barvinok_series_with_options` function and all further `gen_fun` manipulations assume that the effective parameter domain has only revlex-positive rays. When used to computer rational generating functions, the `barvinok_enumerate` application will therefore transform the effective parameter domain of a problem if it has revlex-negative rays. It will then not compute the generating function

$$f(\mathbf{x}) = \sum_{\mathbf{p} \in \mathbb{Z}^m} \#(P_{\mathbf{p}} \cap \mathbb{Z}^d)\, x^{\mathbf{p}},$$

but

$$g(\mathbf{z}) = \sum_{\mathbf{p}' \in \mathbb{Z}^n} \#(P_{T\mathbf{p}'+\mathbf{t}} \cap \mathbb{Z}^d)\, x^{\mathbf{p}'}$$

instead, where $\mathbf{p} = T\mathbf{p}' + \mathbf{t}$, with $T \in \mathbb{Z}^{m \times n}$ and $\mathbf{t} \in \mathbb{Z}^m$, is an affine transformation that maps the transformed parameter space back to the original parameter space.

First assume that the parameter domain does not contain any lines and that there are no equalities in the description of $P_{\mathbf{p}}$ that force the values of $\mathbf{p}$ for which $P_{\mathbf{p}}$ contains integer points to lie on a non-standard lattice. Let the effective parameter domain be given as $\{\, \mathbf{p} \mid A\mathbf{p} + \mathbf{c} \geq \mathbf{0} \,\}$, where $A \in \mathbb{Z}^{s \times d}$ of row rank d; otherwise the effective parameter domain would contain a line. Let H be the (left) HNF of A, i.e.,

$$A = HQ,$$

with H lower-triangular with positive diagonal elements and Q unimodular. Let $\tilde{Q}$ be the matrix obtained from Q by reversing its rows, and, similarly, $\tilde{H}$ from H by reversing the columns. After performing the transformation $\mathbf{p}' = \tilde{Q}\mathbf{p}$, i.e., $\mathbf{p} = \tilde{Q}^{-1}\mathbf{p}'$, the transformed parameter domain is given by

$$\{\, \mathbf{p}' \mid A\tilde{Q}^{-1}\mathbf{p}' + \mathbf{c} \geq \mathbf{0} \,\}$$

or

$$\{\, \mathbf{p}' \mid \tilde{H}\mathbf{p}' + \mathbf{c} \geq \mathbf{0} \,\}.$$

The first constraint of this domain is $h_{11}p'_m + c_1 \geq 0$. A ray with non-zero final coordinate therefore has a positive final coordinate. Similarly, the second constraint is $h_{22}p'_{m-1} + h_{21}p'_m + c_2 \geq 0$. A ray with zero nth coordinate, but non-zero $n-1$st coordinate, will therefore have a positive $n-1$st coordinate. Continuing this reasoning, we see that all rays in the transformed domain are revlex-positive.

If the parameter domain does contains lines, but is not restricted to a non-standard lattice, then the number of points in the parametric polytope is invariant over a translation along the lines. It is therefore sufficient to compute the number of points in the orthogonal complement of the linear subspace spanned by the lines. That is, we apply a prior transformation that maps a reduced parameter domain to this subspace,

$$\mathbf{p} = L^{\perp}\mathbf{p}' = \begin{bmatrix} L & L^{\perp} \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} \mathbf{p}',$$

where L has the lines as columns, and $L^{\perp}$ an integral basis for the orthogonal complement (Section 5.7). Note that the inverse transformation

$$\mathbf{p}' = L^{-\perp}\mathbf{p} = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} L & L^{\perp} \end{bmatrix}^{-1} \mathbf{p}$$

has integral coefficients since $L^{\perp}$ can be extended to a unimodular matrix.

If the parameter values $\mathbf{p}$ for which $P_{\mathbf{p}}$ contains integer points are restricted to a non-standard lattice, we first replace the parameters by a different set of parameters that lie on the standard lattice through “parameter compression”(Meister 2004),

$$\mathbf{p} = C\mathbf{p}'.$$

The (left) inverse of C can be computes as explained in Section 5.6, giving

$$\mathbf{p}' = C^{-L}\mathbf{p}.$$

We have to be careful to only apply this transformation when both the equalities computed in Section 5.6 are satisfied and some additional divisibility constraints. In particular if $\mathbf{a}^{\scriptscriptstyle T}/d$ is a row of C^{-L}, with $\mathbf{a} \in \mathbb{Z}^{n'}$ and $d \in \mathbb{Z}$, the transformation can only be applied to parameter values $\mathbf{p}$ such that d divides $\langle \mathbf{a}, \mathbf{p} \rangle$.

The complete transformation is given by

$$\mathbf{p} = CL^{\perp}\hat{Q}^{-1}\mathbf{p}'$$

or

$$\mathbf{p}' = \hat{Q}L^{-\perp}C^{-L}\mathbf{p}.$$

5.9 Parametric Volume Computation

The volume of a (parametric) polytope can serve as an approximation for the number of integer points in the polytope. We basically follow the description of Rabl (2006) here, except that we focus on volume computation for *linearly* parametrized polytopes, which we exploit to determine the sign of the determinants we compute, as explained below.

Note first that the vertices of a linearly parametrized polytope are affine expressions in the parameters that may be valid only in parts (chambers) of the parameter domain. Since the volume computation is based on the (active) vertices, we perform the computation in each chamber separately. Also note that since the vertices are affine expressions, it is easy to check whether they belong to a facet.

The volume of a d-simplex, i.e., a d-dimensional polytope with $d + 1$ vertices, is relatively easy to compute. In particular, if $\mathbf{v}_i(\mathbf{p})$, for $0 \le i \le d$, are the (parametric) vertices of the simplex P then

$$\operatorname{vol} P = \frac{1}{d!} \left| \det \begin{bmatrix} v_{11}(\mathbf{p}) - v_{01}(\mathbf{p}) & v_{12}(\mathbf{p}) - v_{02}(\mathbf{p}) & \dots & v_{1d}(\mathbf{p}) - v_{0d}(\mathbf{p}) \\ v_{21}(\mathbf{p}) - v_{01}(\mathbf{p}) & v_{22}(\mathbf{p}) - v_{02}(\mathbf{p}) & \dots & v_{2d}(\mathbf{p}) - v_{0d}(\mathbf{p}) \\ \vdots & \vdots & \ddots & \vdots \\ v_{d1}(\mathbf{p}) - v_{01}(\mathbf{p}) & v_{d2}(\mathbf{p}) - v_{02}(\mathbf{p}) & \dots & v_{dd}(\mathbf{p}) - v_{0d}(\mathbf{p}) \end{bmatrix} \right| . \tag{5.25}$$

If P is not a simplex, i.e., $N > d + 1$, with N the number of vertices of P, then the standard way of computing the volume of P is to first *triangulate* P, i.e., subdivide P into simplices, and then to compute and sum the volumes of the resulting simplices. One way of computing a triangulation is to compute the barycenter

$$\frac{1}{N} \sum_i \mathbf{v}_i(\mathbf{p})$$

of P and to perform a subdivision by computing the convex hulls of the barycenter with each of the facets of P. If a given facet of P is itself a simplex, then this convex hull is also a simplex. Otherwise the facet is further subdivided. This recursive process terminates as every 1-dimensional polytope is a simplex.

The triangulation described above is known as the boundary triangulation (Büeler et al. 2000) and is used by Rabl (2006) in his implementation. The Cohen-Hickey triangulation (Cohen and Hickey 1979; Büeler et al. 2000) is a much more efficient variation and uses one of the vertices instead of the barycenter. The facets incident on the vertex do not have to be considered in this case because the resulting subpolytopes would have zero volume. Another possibility is to use a "lifting" triangulation (Lee 1991; De Loera 1995). In this triangulation, each vertex is assigned a (random) "height" in an extra dimension. The projection of the "lower envelope" of the resulting polytope onto the original space results in a subdivision, which is a triangulation with very high probability.

A complication with the lifting triangulation is that the constraint system of the lifted polytope will in general not be linearly parameterized, even if the original polytope is. It is, however, sufficient to perform the triangulation for a particular value of the parameters inside the chamber since the parametric polytope has the same combinatorial structure throughout the chamber. The triangulation obtained for the instantiated vertices can then be carried over to the corresponding parametric vertices. We only need to be careful to select a value for the parameters that does not lie on any facet of the chambers. On these chambers, some of the vertices may coincide. For linearly parametrized polytopes, it is easy to find a parameter point in the interior of a chamber, as explained in Section 5.1. Note that this point need not be integer.

A direct application of the above algorithm, using any of the triangulations, would yield for each chamber a volume expressed as the sum of the absolute values of polynomials in the parameters. To remove the absolute value, we plug in a particular value of the parameters (not necessarily integer) belonging to the given chamber for which we know that the volume is non-zero. Again, it is sufficient to take any point in the interior of the chamber. The sign of the resulting value then determines the sign of the whole polynomial since polynomials are continuous functions and will not change sign without passing through zero.

5.10 Maclaurin series division

If $P(t)$ and $Q(t)$ are two Maclaurin series

$$\begin{aligned} P(t) &= a_0 + a_1 t + a_2 t^2 + \cdots \\ Q(t) &= b_0 + b_1 t + b_2 t^2 + \cdots , \end{aligned}$$

then, as outlined by Henrici (1974, 241–247), we can compute the coefficients c_l in

$$\frac{P(t)}{Q(t)} =: c_0 + c_1 t + c_2 t^2 + \cdots$$

by applying the recurrence relation

$$c_l = \frac{1}{b_0}\left(a_l - \sum_{i=1}^{l} b_i c_{l-i} \right).$$

To avoid dealing with denominators, we can also compute $d_l = b_0^{l+1} c_l$ instead as

$$d_l = b_0^l a_l - \sum_{i=1}^{l} b_0^{i-1} b_i c_{l-i}.$$

The coefficients c_l can then be directly read off as

$$c_l = \frac{d_l}{b_0^{l+1}}.$$

5.11 Specialization through exponential substitution

This section draws heavily from De Loera and Köppe (2006).

We define a “short” *rational generating function* to be a function of the form

$$f(\mathbf{x}) = \sum_{i\in I} \alpha_i \frac{\sum_{k=1}^{r} \mathbf{x}^{\mathbf{w}_{ik}}}{\prod_{j=1}^{k_i} (1 - \mathbf{x}^{\mathbf{b}_{ij}})}, \tag{5.26}$$

with $\mathbf{x} \in \mathbb{C}^d$, $\alpha_i \in \mathbb{Q}$, $\mathbf{w}_{ik} \in \mathbb{Z}^d$ and $\mathbf{b}_{ij} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$.

After computing the rational generating function (5.26) of a polytope (with $k_i = d$ for all i), the number of lattice points in the polytope can be obtained by evaluating $f(\mathbf{1})$. Since $\mathbf{1}$ is a pole of each term, we need to compute the constant term in the Laurent expansions of each term in (5.26) about $\mathbf{1}$. Since it is easier to work with univariate series, a substitution is usually applied, either a polynomial substitution

$$\mathbf{x} = (1+t)^{\boldsymbol{\lambda}},$$

as implemented in `LattE` (De Loera et al. 2003), or an exponential substitution (see, e.g., Barvinok and Pommersheim 1999),

$$\mathbf{x} = e^{t\boldsymbol{\lambda}},$$

as implemented in `LattE macchiato` (Köppe 2006). In each case, $\boldsymbol{\lambda} \in \mathbb{Z}^d$ is a vector that is not orthogonal to any of the $\mathbf{b}_{ij}$. Both substitutions also transform the problem of computing the constant term in the Laurent expansions about $\mathbf{x} = \mathbf{1}$ to that of computing the constant term in the Laurent expansions about $t = 0$. Here, we discuss the exponential substitution.

Consider now one of the terms in (5.26),

$$g(t) = \frac{\sum_{k=1}^{r} e^{a_k t}}{\prod_{j=1}^{d} (1 - e^{c_j t})},$$

with $a_k = \langle \mathbf{w}_{ik}, \boldsymbol{\lambda} \rangle$ and $c_j = \langle \mathbf{b}_{ij}, \boldsymbol{\lambda} \rangle$. We rewrite this equation as

$$g(t) = (-1)^d \frac{\sum_{k=1}^{r} e^{a_k t}}{t^d \prod_{j=1}^{d} c_j} \prod_{j=1}^{d} \frac{-c_j t}{1 - e^{c_j t}}.$$

The second factor is analytic in a neighborhood of the origin $t = c_1 = \cdots = c_d = 0$ and therefore has a Taylor series expansion

$$\prod_{j=1}^{d} \frac{-c_j t}{1 - e^{c_j t}} = \sum_{m=0}^{\infty} \mathrm{td}_m(-c_1, \ldots, -c_d) t^m, \tag{5.27}$$

where td_m is a homogeneous polynomial of degree m called the m-th Todd polynomial (Barvinok and Pommersheim 1999). Also expanding the numerator in the first factor, we find

$$g(t) = \frac{(-1)^d}{t^d \prod_{j=1}^{d} c_j} \left(\sum_{n=0}^{\infty} \frac{\sum_{k=1}^{r} a_k^n}{n!} t^n \right) \left(\sum_{m=0}^{\infty} \mathrm{td}_m(-c_1, \ldots, -c_d) t^m \right),$$

with constant term

$$\frac{(-1)^d}{t^d \prod_{j=1}^d c_j} \left(\sum_{i=0}^d \frac{\sum_{k=1}^r a_k^i}{i!} \operatorname{td}_{d-i}(-c_1, \ldots, -c_d) \right) t^d = \frac{(-1)^d}{\prod_{j=1}^d c_j} \sum_{i=0}^d \frac{\sum_{k=1}^r a_k^i}{i!} \operatorname{td}_{d-i}(-c_1, \ldots, -c_d). \tag{5.28}$$

To compute the first $d+1$ terms in the Taylor series (5.27), we write down the truncated Taylor series

$$\frac{e^t - 1}{t} \equiv \sum_{i=0}^d \frac{1}{(i+1)!} t^i \equiv \frac{1}{(d+1)!} \sum_{i=0}^d \frac{(d+1)!}{(i+1)!} t^i \mod t^{d+1},$$

where we have

$$\frac{1}{(d+1)!} \sum_{i=0}^d \frac{(d+1)!}{(i+1)!} t^i \in \frac{1}{(d+1)!} \mathbb{Z}[t].$$

Computing the reciprocal as explained in Section 5.10, we find

$$\frac{-t}{1-e^t} = \frac{t}{e^t - 1} = \frac{1}{\frac{e^t-1}{t}} \equiv (d+1)! \frac{1}{\sum_{i=0}^d \frac{(d+1)!}{(i+1)!} t^i} =: \sum_{i=0}^d b_i t^i. \tag{5.29}$$

Note that the constant term of the denominator is $1/(d+1)!$. The denominators of the quotient are therefore $((d+1)!)^{i+1}/(d+1)!$. Also note that the b_i are independent of the generating function and can be computed in advance. An alternative way of computing the b_i is to note that

$$\frac{t}{e^t - 1} = \sum_{i=0}^\infty B_i \frac{t^i}{i!},$$

with $B_i = i!\, b_i$ the Bernoulli numbers, which can be computed using the recurrence (5.34) (see Section 5.12).

Substituting t by $c_j t$ in (5.29), we have

$$\frac{-c_j t}{1 - e^{c_j t}} = \sum_{i=0}^d b_i c_j^i t^i.$$

Multiplication of these truncated Taylor series for each c_j results in the first $d+1$ terms of (5.27),

$$\sum_{m=0}^d \operatorname{td}_m(-c_1, \ldots, -c_d) t^m =: \sum_{m=0}^d \frac{\beta_m}{((d+1)!)^m} t^m,$$

from which it is easy to compute the constant term (5.28). Note that this convolution can also be computed without the use of rational coefficients,

$$\frac{(-1)^d}{\prod_{j=1}^d c_j} \sum_{i=0}^d \frac{\alpha_i}{i!} \frac{\beta_{d-i}}{((d+1)!)^{d-i}} = \frac{(-1)^d}{((d+1)!)^d \prod_{j=1}^d c_j} \sum_{i=0}^d \left(\frac{((d+1)!)^i}{i!} \alpha_i \right) \beta_{d-i},$$

with $\alpha_i = \sum_{k=1}^r a_k^i$.

Example 5.30 *Consider the rational generating function*

$$f(T;\mathbf{x}) = \frac{x_1^2}{(1-x_1^{-1})(1-x_1^{-1}x_2)} + \frac{x_2^2}{(1-x_2^{-1})(1-x_1x_2^{-1})} + \frac{1}{(1-x_1)(1-x_2)}$$

from Verdoolaege (2005, Example 39). Since this is a 2-dimensional problem, we first compute the first 3 Todd polynomials (evaluated at -1),

$$\frac{e^t-1}{t} \equiv 1 + \frac{1}{2}t + \frac{1}{6}t^2 = \frac{1}{6}\begin{bmatrix} 6 & 3 & 1 \end{bmatrix}$$

and

$$\frac{-t}{1-e^t} = \frac{t}{e^t-1} \equiv \begin{bmatrix} \frac{1}{1} & \frac{-3}{6} & \frac{3}{36} \end{bmatrix},$$

where we represent each truncated power series by a vector of its coefficients. The vector $\boldsymbol{\lambda} = (1,-1)$ is not orthogonal to any of the rays, so we can use the substitution $\mathbf{x} = e^{(1,-1)t}$ and obtain

$$\frac{e^{2t}}{(1-e^{-t})(1-e^{-2t})} + \frac{e^{-2t}}{(1-e^{t})(1-e^{2t})} + \frac{1}{(1-e^{t})(1-e^{-t})}.$$

We have

$$\begin{aligned}
\frac{t}{1-e^{-t}} &= \begin{bmatrix} \frac{1}{1} & \frac{3}{6} & \frac{3}{36} \end{bmatrix}\\
\frac{2t}{1-e^{-2t}} &= \begin{bmatrix} \frac{1}{1} & \frac{6}{6} & \frac{12}{36} \end{bmatrix}\\
\frac{-t}{1-e^{t}} &= \begin{bmatrix} \frac{1}{1} & \frac{-3}{6} & \frac{3}{36} \end{bmatrix}\\
\frac{-2t}{1-e^{2t}} &= \begin{bmatrix} \frac{1}{1} & \frac{-6}{6} & \frac{12}{36} \end{bmatrix}.
\end{aligned}$$

The first term in the rational generating function evaluates to

$$\begin{aligned}
&\frac{1}{-1\cdot -2}\begin{bmatrix} \frac{1}{1} & \frac{2}{1} & \frac{4}{2} \end{bmatrix} * \left(\begin{bmatrix} \frac{1}{1} & \frac{3}{6} & \frac{3}{36} \end{bmatrix}\begin{bmatrix} \frac{1}{1} & \frac{6}{6} & \frac{12}{36} \end{bmatrix}\right)\\
&= \frac{1}{2}\begin{bmatrix} \frac{1}{1} & \frac{2}{1} & \frac{4}{2} \end{bmatrix} * \begin{bmatrix} \frac{1}{1} & \frac{9}{6} & \frac{33}{36} \end{bmatrix}\\
&= \frac{1}{72}\begin{bmatrix} 1 & 2\cdot 6 & 4\cdot 18 \end{bmatrix} * \begin{bmatrix} 1 & 9 & 33 \end{bmatrix} = \frac{213}{72} = \frac{71}{24}.
\end{aligned}$$

Due to symmetry, the second term evaluates to the same value, while for the third term we find

$$\frac{1}{-1\cdot 1\cdot 36}\begin{bmatrix} 1 & 0\cdot 6 & 0\cdot 18 \end{bmatrix} * \begin{bmatrix} 1 & 0 & -3 \end{bmatrix} = \frac{-3}{-36} = \frac{1}{12}.$$

The sum is

$$\frac{71}{24} + \frac{71}{24} + \frac{1}{12} = 6.$$

Note that the run-time complexities of polynomial and exponential substitution are basically the same. The experiments of Köppe (2007) are somewhat misleading in this respect since the polynomial substitution (unlike the exponential substitution) had not been optimized to take full advantage of the stopped Barvinok decomposition. For comparison, Table 1 shows running times for the same experiments of that paper, but using barvinok version `barvinok-0.23-47-gaa9024e` on an Athlon MP 1500+ with 512MiB internal memory. This machine appears to be slightly slower than the machine used in the experiments of Köppe (2007) as computing `hickerson-14` using the dual decomposition with polynomial substitution and maximal index 1 took 2768 seconds on this machine using `LattE macchiato`. At this stage, it is not clear yet why the number of cones in the dual decomposition of `hickerson-13` differs from that of `LattE` (De Loera et al. 2003) and `LattE macchiato` (Köppe 2006). We conclude from Table 1 that (our implementation of) the exponential substitution is always slightly faster than (our implementation of) the polynomial substitution. The optimal maximal index for these examples is about 500, which agrees with the experiments of Köppe (2007).

5.12 Approximate Enumeration using Nested Sums

If $P \in \mathbb{Q}^d$ is a polyhedron and $p(\mathbf{x}) \in \mathbb{Q}[\mathbf{x}]$ is a polynomial and we want to sum $p(\mathbf{x})$ over all integer values of (a subset of) the variables $\mathbf{x}$, then we can do this incrementally by taking a variable x_1 with lower bound $L(\hat{\mathbf{x}})$ and upper bound $U(\hat{\mathbf{x}})$, with $\hat{\mathbf{x}} = (x_2, \ldots, x_d)$, and computing

$$Q(\hat{\mathbf{x}}) = \sum_{x_1=L(\hat{\mathbf{x}})}^{U(\hat{\mathbf{x}})} p(\mathbf{x}). \tag{5.31}$$

Since P is a polytope, the lower bound is a maximum of affine expressions in the remaining variables, while the upper bound is a minimum of such expressions. If the coefficients in these expressions are all integer, then we can compute $Q(\hat{\mathbf{x}})$ exactly as a piecewise polynomial using formulas for sums of powers, as proposed by, e.g., Tawbi (1994), Sakellariou (1997), Van Engelen et al. (2006). If some of the coefficients are not integer, we can apply the same formulas to obtain an approximation, which can is some cases be shown to be an overapproximation (Van Engelen et al. 2006). Note that if we take the initial polynomial to be the constant 1, then this gives us a method for computing an approximation of the number of integer points in a (parametric) polytope.

The first step is to compute the chamber decomposition of P when viewed as a 1-dimensional parametric polytope. That is, we need to partition the projection of P onto the remaining variables into polyhedral cells such that in each cell, both the upper and the lower bound are described by a single affine expression. Basically, for each pair of lower and upper bound, we compute the cell where the chosen lower bound is (strictly) smaller than all other lower bounds and similarly for the upper bound.

For any given pair of lower and upper bound $(l(\hat{\mathbf{x}}), u(\hat{\mathbf{x}}))$, the formula (5.31) is computed for each monomial of $p(\mathbf{x})$ separately. For the constant term α_0, we have

	Dual decomposition			Primal decomposition		
		Time (s)			Time (s)	
Max. index	Cones	Poly	Exp	Cones	Poly	Exp
			`hickerson-12`			
1	11625	9.24	8.90	7929	4.80	4.55
10	4251	4.32	4.19	803	0.66	0.62
100	980	1.42	1.35	84	0.13	0.12
200	550	1.00	0.92	76	0.12	0.12
300	474	0.93	0.86	58	0.12	0.10
500	410	0.90	0.83	42	0.10	0.10
1000	130	0.42	0.38	22	**0.10**	**0.07**
2000	10	**0.10**	**0.10**	22	0.10	0.09
5000	7	0.12	0.11	7	0.12	0.10
			`hickerson-13`			
1	494836	489	463	483507	339	315
10	296151	325	309	55643	51	48
100	158929	203	192	9158	11	10
200	138296	184	173	6150	9	8
300	110438	168	157	4674	8	7
500	102403	163	151	3381	**8**	**7**
1000	83421	**163**	**149**	2490	8	7
2000	77055	170	153	1857	10	8
5000	57265	246	211	1488	13	11
10000	50963	319	269	1011	26	21
			`hickerson-14`			
1	1682743	2171	2064	552065	508	475
10	1027619	1453	1385	49632	62	59
100	455474	768	730	8470	14	13
200	406491	699	661	5554	11	10
300	328340	627	590	4332	11	9
500	303566	605	565	3464	**11**	**9**
1000	232626	**581**	**532**	2384	12	10
2000	195368	607	545	1792	14	12
5000	147496	785	682	1276	19	16
10000	128372	966	824	956	29	23

Table 1: Timing results of dual and primal decomposition with polynomial or exponential substitution on the Hickerson examples

$$\sum_{x_1=l(\hat{\mathbf{x}})}^{u(\hat{\mathbf{x}})} \alpha_0(\hat{\mathbf{x}}) = \alpha_0(\hat{\mathbf{x}}) \left(u(\hat{\mathbf{x}}) - l(\hat{\mathbf{x}}) + 1\right). \tag{5.32}$$

For the higher degree monomials, we use the formula

$$\sum_{k=0}^{m-1} k^n = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k m^{n+1-k} =: S_n(m), \tag{5.33}$$

with B_i the Bernoulli numbers, which can be computed using the recurrence

$$\sum_{j=0}^{m} \binom{m+1}{j} B_j = 0 \qquad B_0 = 1. \tag{5.34}$$

Note that (5.33) is also valid if $m = 0$, i.e., $S_n(0) = 0$, a fact that can be easily shown using Newton series (Van Engelen et al. 2006).

Since we can only directly apply the summation formula when the lower bound is zero (or one), we need to consider several cases.

1. $l(\hat{\mathbf{x}}) \geq 1$

$$\begin{aligned}\sum_{x_1=l(\hat{\mathbf{x}})}^{u(\hat{\mathbf{x}})} \alpha_n(\hat{\mathbf{x}})\, x_1^n &= \alpha_n(\hat{\mathbf{x}}) \left(\sum_{x_1=1}^{u(\hat{\mathbf{x}})} x_1^n - \sum_{x_1=1}^{l(\hat{\mathbf{x}})-1} x_1^n \right) \\ &= \alpha_n(\hat{\mathbf{x}}) \left(S_n(u(\hat{\mathbf{x}})+1) - S_n(l(\hat{\mathbf{x}}))\right)\end{aligned}$$

2. $u(\hat{\mathbf{x}}) \leq -1$

$$\begin{aligned}\sum_{x_1=l(\hat{\mathbf{x}})}^{u(\hat{\mathbf{x}})} \alpha_n(\hat{\mathbf{x}})\, x_1^n &= \alpha_n(\hat{\mathbf{x}})(-1)^n \sum_{x_1=-u(\hat{\mathbf{x}})}^{-l(\hat{\mathbf{x}})} \alpha_n(\hat{\mathbf{x}})\, x_1^n \\ &= \alpha_n(\hat{\mathbf{x}})(-1)^n \left(S_n(-l(\hat{\mathbf{x}})+1) - S_n(-u(\hat{\mathbf{x}}))\right)\end{aligned}$$

3. $l(\hat{\mathbf{x}}) \leq 0$ and $u(\hat{\mathbf{x}}) \geq 0$

$$\begin{aligned}\sum_{x_1=l(\hat{\mathbf{x}})}^{u(\hat{\mathbf{x}})} \alpha_n(\hat{\mathbf{x}})\, x_1^n &= \alpha_n(\hat{\mathbf{x}}) \left(\sum_{x_1=0}^{u(\hat{\mathbf{x}})} x_1^n + (-1)^n \sum_{x_1=1}^{-l(\hat{\mathbf{x}})} x_1^n \right) \\ &= \alpha_n(\hat{\mathbf{x}}) \left(S_n(u(\hat{\mathbf{x}})+1) + (-1)^n S_n(-l(\hat{\mathbf{x}})+1)\right)\end{aligned}$$

If the coefficients in the lower and upper bound are all integer, then the above 3 cases partition (the integer points in) the projection of P onto the remaining variables. However, if some of the coefficients are rational, then the lower and upper bound can lie in the open interval $(0, 1)$ for some values of $\hat{\mathbf{x}}$. We may therefore also want to consider the following two cases.

4. $0 < l(\hat{\mathbf{x}}) < 1$

$$\sum_{x_1=l(\hat{\mathbf{x}})}^{u(\hat{\mathbf{x}})} \alpha_n(\hat{\mathbf{x}})\, x_1^n = \alpha_n(\hat{\mathbf{x}}) S_n(u(\hat{\mathbf{x}}) + 1)$$

5. $0 < -u(\hat{\mathbf{x}}) < 1$

$$\sum_{x_1=l(\hat{\mathbf{x}})}^{u(\hat{\mathbf{x}})} \alpha_n(\hat{\mathbf{x}})\, x_1^n = \alpha_n(\hat{\mathbf{x}})(-1)^n S_n(-l(\hat{\mathbf{x}}) + 1)$$

Note that we may add the constraint $u \geq 1$ to case 4 and the constraint $l \leq -1$ to case 5, since the correct value for these two cases would be zero if these extra constraints do not hold.

An alternative to adding the above two cases would be to simply ignore them, i.e., assume a value of 0. Another alternative would be to reduce case 3 to

$$l(\hat{\mathbf{x}}) \leq -1 \quad \text{and} \quad u(\hat{\mathbf{x}}) \geq 1,$$

while extending cases 4 and 5 to

$$-1 < l(\hat{\mathbf{x}}) < 1 \quad \text{and} \quad u \geq 1$$

and

$$-1 < u(\hat{\mathbf{x}}) < 1 \quad \text{and} \quad l \leq -1,$$

respectively, with the remaining cases ($-1 < l \leq u < 1$) having value 0. There does not appear to be a consistently better choice here, as each of these three approaches seems to yield better results on some examples. The last approach has the additional drawback that we would also have to deal with 5 cases, even if the bounds are integers.

If at least one of the lower or upper bound is an integer affine expression, then we can reduce the 3 (or 5) cases to a single case (case 3) by an affine substitution that ensure that the new (lower or upper) bound is zero. In particular, if $l(\hat{\mathbf{x}})$ is an integer affine expression, then we replace x by $x' + l(\hat{\mathbf{x}})$ and similarly for an upper bound.

5.13 Exact Enumeration using Nested Sums

The exact enumeration using nested sums proceeds in much the same way as the approximate enumeration from subsection 5.12, with the notable exception that we need to take the (greatest or least) integer part of any fractional bounds that may occur. This has several consequences, discussed below.

Since we will introduce floors during the recursive application of the procedure, we may as well allow the weight $p(\mathbf{x})$ in (5.31) to be a (piecewise) quasipolynomial.

For the constant term, (5.32) becomes

$$\sum_{x_1=l(\hat{\mathbf{x}})}^{u(\hat{\mathbf{x}})} \alpha_0(\hat{\mathbf{x}}) = \alpha_0(\hat{\mathbf{x}}) \left(\lfloor u(\hat{\mathbf{x}}) \rfloor - \lceil l(\hat{\mathbf{x}}) \rceil + 1\right).$$

Since we force the lower and upper bounds to be integers, cases 4 and 5 do not occur, while the conditions for cases 1 and 2 can be simplified to

$$l(\hat{\mathbf{x}}) > 0$$

and

$$u(\hat{\mathbf{x}}) < 0,$$

respectively.

If the variable x appears in any floor expression, either because such an expression was present in the original weight function or because it was introduced when another variable with an affine bound in x was summed, then the domain has to be "splintered" into D parts, where D is the least common multiple of the denominators of the coefficients of x in any of the integer parts. In particular, the domain is split into $x = Dy + i$ for each i in $[0, D - 1]$. Since D is proportional to the coefficients in the constraints, it is exponential in the input size. This splintering will therefore introduce exponential behavior, even if the dimension is fixed.

Splintering is clearly the most expensive step in the algorithm, so we want to avoid this step as much as possible. Pugh (1994) already noted that summation should proceed over variables with integer bounds first. This can be extended to choosing a variable with the smallest coefficient in absolute value. In this way, we can avoid splintering on the largest denominator.

Sakellariou (1996) claims that splintering can be avoided altogether. In particular, Sakellariou (1996, Lemma 3.2) shows that

$$\sum_{x=0}^{a} x^m \left(x \bmod b\right)^n ,$$

with a and b integers, is equal to

$$\begin{cases} \displaystyle\sum_{x=0}^{a} x^{m+n} & \text{if } a < b \\ \displaystyle\sum_{i=0}^{\lfloor a/b \rfloor - 1} \sum_{x=0}^{b-1} (x + ib)^m x^n + \sum_{x=0}^{a \bmod b} (x + b \lfloor a/b \rfloor)^m x^n & \text{if } a \geq b, \end{cases} \tag{5.35}$$

effectively avoiding splintering if a given monomial contains a single integer part expression with argument of the form x/b. An argument of the form $(x - c(\hat{\mathbf{x}}))/b$ can be handled through a variable substitution. If the argument is of the form cx/b, with $c \neq 1$, then Sakellariou (1996, (3.27)) proposes to rewrite the monomial as

$$\begin{aligned} \sum_{x=0}^{a} (cx \bmod b)^n &= \sum_{x=0}^{a} \sum_{y=cx}^{cx} (y \bmod b)^n \\ &= \sum_{x=0}^{a} \left(\sum_{y=0}^{cx} (y \bmod b)^n - \sum_{y=0}^{cx-1} (y \bmod b)^n \right) \end{aligned}$$

and applying (5.35). However, such an application results in an expression containing

$$\sum_{y=0}^{cx \bmod b} y^n,$$

which in turn leads to a polynomial of degree $n + 1$ in $(cx \bmod b)$, i.e., of degree one higher than the original expression. Furthermore, if the bound on x is rational then a itself contains a floor, which, on application of (5.35), results in a nested floor expression, blocking the application of the same rule for the next variable. Finally, the case where a monomial contains multiple floor expressions, either occurring in the input quasi-polynomial or introduced by different variables having a rational bound with a non-zero coefficient in the same variable, is not handled. Also note that if we disallow nested floor expressions, then this rule will rarely be applicable since we try to eliminate variables with integer bounds first.

5.14 Summation using local Euler-Maclaurin formula

In this section we provide some implementation details on using local Euler-Maclaurin formula to compute the sum of a piecewise polynomial evaluated in all integer points of a two-dimensional parametric polytope. For the theory behind these formula and a discussion of the original implementation (for non-parametric simplices), we refer to Berline and Vergne (2006).

In particular, consider a parametric piecewise polynomial in n parameters and m variables $c : \mathbb{Z}^n \to \mathbb{Z}^m \to \mathbb{Q} : \mathbf{p} \mapsto c(\mathbf{p})$, with $c(\mathbf{p}) : \mathbb{Z}^m \to \mathbb{Q} : \mathbf{x} \mapsto c(\mathbf{p})(\mathbf{x})$ and

$$c_{\mathbf{p}}(\mathbf{x}) = \begin{cases} c_1(\mathbf{p})(\mathbf{x}) & \text{if } \mathbf{x} \in D_1(\mathbf{p}) \\ \vdots \\ c_r(\mathbf{p})(\mathbf{x}) & \text{if } \mathbf{x} \in D_r(\mathbf{p}), \end{cases}$$

with the c_i polynomials, $c_i \in (\mathbb{Q}[\mathbf{p}])[\mathbf{x}]$, and the D_i disjoint linearly parametric polytopes. We want to compute

$$g(\mathbf{p}) = \sum_{\mathbf{x} \in \mathbb{Z}^m} c(\mathbf{p})(\mathbf{x}).$$

5.14.1 Reduction to the summation of a parametric polynomial over a parametric polytope with a fixed combinatorial structure

Since the D_i are disjoint, we can consider each (c_i, D_i)-pair individually and compute

$$g(\mathbf{p}) = \sum_{i=1}^{r} g_i(\mathbf{p}) = \sum_{i=1}^{r} \sum_{\mathbf{x} \in D_r(\mathbf{p}) \cap \mathbb{Z}^m} c_r(\mathbf{p})(\mathbf{x}).$$

The second step is to compute the chamber decomposition (Verdoolaege 2005, Section 4.2.3) of each parametric polytope D_i. The result is a subdivision of the parameter space into chambers C_{ij} such that D_i has a fixed combinatorial structure, in particular a fixed set of parametric vertices, on (the interior of) each C_{ij}. Applying Theorem 5.12, this subdivision can be transformed into a partition $\{\tilde{C}_{ij}\}$ by making some of the facets of the chambers open (Köppe and Verdoolaege 2008, Section 3.2). Since we are only interested in integer parameter values, any of the resulting open facets $\langle \mathbf{a}, \mathbf{p} \rangle + c > 0$, with $\mathbf{a} \in \mathbb{Z}^n$ and $c \in \mathbb{Z}$, can then be replaced by $\langle \mathbf{a}, \mathbf{p} \rangle + c - 1 \geq 0$. Again, we have

$$g_i(\mathbf{p}) = \sum_j g_{ij}(\mathbf{p}) = \sum_j \sum_{\mathbf{x} \in C_{ij}(\mathbf{p}) \cap \mathbb{Z}^m} c_r(\mathbf{p})(\mathbf{x}).$$

After this reduction, the technique of Berline and Vergne (2006) can be applied practically verbatim to the parametric polytope with a fixed combinatorial structure. In principle, we could also handle piecewise quasi-polynomials using the technique of Verdoolaege (2005, Section 4.5.4), except that we only need to create an extra variable for each distinct floor expression in a monomial, rather than for each occurrence of a floor expression in a monomial. However, since we currently only support two-dimensional polytopes, this reduction has not been implemented yet.

5.14.2 Summation over a one-dimensional parametric polytope

The basis for the summation technique is the local Euler-Maclaurin formula (Berline and Vergne 2006, Theorem 26)

$$\sum_{\mathbf{x} \in P(\mathbf{p}) \cap \Lambda} h(\mathbf{p})(\mathbf{x}) = \sum_{F(\mathbf{p}) \in \mathcal{F}(P(\mathbf{p}))} \int_{F(\mathbf{p})} D_{P(\mathbf{p}),F(\mathbf{p})} \cdot h(\mathbf{p}), \tag{5.36}$$

where $P(\mathbf{p})$ is a parametric polytope, Λ is a lattice, $\mathcal{F}(P(\mathbf{p}))$ are the faces of $P(\mathbf{p})$, $D_{P(\mathbf{p}),F(\mathbf{p})}$ is a specific differential operator associated to the face of a polytope. The Lebesgue measure used in the integral is such that the integral of the indicator function of a lattice element of the lattice $\Lambda \cap (\operatorname{aff}(F(\mathbf{p})) - F(\mathbf{p}))$ is 1, i.e., the intersection of Λ with the linear subspace parallel to the affine hull of the face $F(\mathbf{p})$. Note that the original theorem is formulated for a non-parametric polytope and a non-parametric polynomial. However, as we will see, in each of the steps in the computation, the parameters can be treated as symbolic constants without affecting the validity of the formula, see also Berline and Vergne (2006, Section 6).

The differential operator $D_{P(\mathbf{p}),F(\mathbf{p})}$ is obtained by plugging in the vector $\mathbf{D} = (D_1, \ldots, D_m)$ of first order differential operators, i.e., D_k is the first order differential operator in the kth variable, in the function $\mu_{P(\mathbf{p}),F(\mathbf{p})}$. This function is determined by the *transverse cone* of the polyhedron $P(\mathbf{p})$ along its face $F(\mathbf{p})$, which is the supporting cone of $P(\mathbf{p})$ along $F(\mathbf{p})$ projected into the linear subspace orthogonal to $F(\mathbf{p})$. The lattice associated to this space is the projection of Λ into this space.

In particular, for a zero-dimensional affine cone in the zero-dimensional space, we have $\mu = 1$ (Berline and Vergne 2006, Proposition 12), while for a one-dimensional affine cone $K = (-t + \mathbb{R}_+)r$ in the one-dimensional space, where r is a primitive integer vector and $t \in [0, 1)$, we have (Berline and Vergne 2006, (13))

$$\mu(K)(\xi) = \frac{e^{ty}}{1 - e^y} + \frac{1}{y} = -\sum_{n=0}^{\infty} \frac{b(n+1,t)}{(n+1)!} y^n, \tag{5.37}$$

with $y = \langle \xi, r \rangle$ and $b(n,t)$ the Bernoulli polynomials defined by the generating series

$$\operatorname{Todd}(t, y) = \frac{e^{ty} y}{e^y - 1} = \sum_{n=0}^{\infty} \frac{b(n,t)}{n!} y^n. \tag{5.38}$$

The constant terms of these Bernoulli polynomials are the Bernoulli numbers.

Applying (5.36) to a one-dimensional parametric polytope $P(\mathbf{p}) = [v_1(\mathbf{p}), v_2(\mathbf{p})]$, we find

$$\begin{aligned}\sum_{x \in P(\mathbf{p}) \cap \mathbb{Z}} h(\mathbf{p})(x) = \int_{P(\mathbf{p})} & D_{P(\mathbf{p}),P(\mathbf{p})} \cdot h(\mathbf{p}) \\ & + \int_{v_1(\mathbf{p})} D_{P(\mathbf{p}),v_1(\mathbf{p})} \cdot h(\mathbf{p}) \\ & + \int_{v_2(\mathbf{p})} D_{P(\mathbf{p}),v_2(\mathbf{p})} \cdot h(\mathbf{p}).\end{aligned}$$

The transverse cone of a polytope along the whole polytope is a zero-dimensional cone in a zero-dimensional space and so $D_{P(\mathbf{p}),P(\mathbf{p})} = \mu_{P(\mathbf{p}),P(\mathbf{p})}(D) = 1$. The transverse cone along $v_1(\mathbf{p})$ is $v_1(\mathbf{p}) + \mathbb{R}_+$ and so $D_{P(\mathbf{p}),v_1(\mathbf{p})} = \mu(v_1(\mathbf{p}) + \mathbb{R}_+)(D)$ as in (5.37), with $y = \langle D, 1 \rangle = D$ and $t = \lceil v_1(\mathbf{p}) \rceil - v_1(\mathbf{p}) = \{-v_1(\mathbf{p})\}$. Similarly we find $D_{P(\mathbf{p}),v_2(\mathbf{p})} = \mu(v_2(\mathbf{p}) - \mathbb{R}_+)(D)$ as in (5.37), with $y = \langle D, -1 \rangle = -D$ and $t = v_2(\mathbf{p}) - \lfloor v_2(\mathbf{p}) \rfloor = \{v_2(\mathbf{p})\}$. Summarizing, we find

$$\begin{aligned}\sum_{x \in P(\mathbf{p}) \cap \mathbb{Z}} h(\mathbf{p})(x) = \int_{v_1(\mathbf{p})}^{v_2(\mathbf{p})} & h(\mathbf{p})(t)\, dt \\ & - \sum_{n=0}^{\infty} \frac{b(n+1, \{-v_1(\mathbf{p})\})}{(n+1)!} (D^n h(\mathbf{p}))(v_1(\mathbf{p})) \\ & - \sum_{n=0}^{\infty} (-1)^n \frac{b(n+1, \{v_2(\mathbf{p})\})}{(n+1)!} (D^n h(\mathbf{p}))(v_2(\mathbf{p})).\end{aligned}$$

Note that in order to apply this formula, we need to verify first that $v_1(\mathbf{p})$ is indeed smaller than (or equal to) $v_2(\mathbf{p})$. Since the combinatorial structure of $P(\mathbf{p})$ does not change throughout the interior of the chamber, we only need to check the order of the two vertices for one value of the parameters from the interior of the chamber, a point which we may compute as in subsection 5.1.

5.14.3 Summation over a two-dimensional parametric polytope

For two-dimensional polytope, formula (5.36) has three kinds of contributions: the integral of the polynomial over the polytope, contributions along edges and contributions along vertices. As suggested by Berline (2007), the integral can be computed by applying the Green-Stokes theorem:

$$\iint_{P(\mathbf{p})} \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) = \int_{\partial P(\mathbf{p})} (L\, dx + M\, dy).$$

In particular, if $M(\mathbf{p})(x,y)$ is such that $\frac{\partial M}{\partial x}(\mathbf{p})(x,y) = h(\mathbf{p})(x,y)$ then

$$\iint_{P(\mathbf{p})} h(\mathbf{p})(x,y) = \int_{\partial P(\mathbf{p})} M(\mathbf{p})(x,y)\, dy.$$

Care must be taken to integrate over the boundary in the positive direction. Assuming the vertices of the polygon are not given in a predetermined order, we can check the correct orientation of the vertices of each edge individually. Let $\mathbf{n} = (n_1, n_2)$ be the inner normal of a facet and let $\mathbf{v}_1(\mathbf{p})$ and $\mathbf{v}_2(\mathbf{p})$ be the two vertices of the facet, then the vertices are in the correct order if

$$\begin{vmatrix} v_{2,1}(\mathbf{p}) - v_{1,1}(\mathbf{p}) & n_1 \\ v_{2,2}(\mathbf{p}) - v_{1,2}(\mathbf{p}) & n_2 \end{vmatrix} \geq 0.$$

Since these two vertices belong to the same edge, their order will not change within a chamber and so we can again perform this check for a single value of the parameters. To integrate M over an edge F, let $\mathbf{f}$ be a primitive integer vector in the direction of the edge. Then $\mathbf{v}_2(\mathbf{p}) = \mathbf{v}_1(\mathbf{p}) + k(\mathbf{p})\,\mathbf{f}$ and any point on the edge can be written as $\mathbf{v}_1(\mathbf{p}) + \lambda\mathbf{f}$ with $0 \leq \lambda \leq k(\mathbf{p})$. That is,

$$\int_F M(\mathbf{p})(x,y)\, dy = \int_0^{k(\mathbf{p})} M(\mathbf{p})(v_{1,1}(\mathbf{p}) + \lambda f_1, v_{1,2}(\mathbf{p}) + \lambda f_2) f_2\, d\lambda. \tag{5.39}$$

For the edges, we can again apply (5.37), but we must first project the supporting cone at the edge into the linear subspace orthogonal to the edge. Let $\mathbf{n} = (n_1, n_2)$ be the (primitive integer) inner normal of this facet $F(\mathbf{p})$, then $\mathbf{f} = (-n_2, n_1)$ is parallel to the facet and we can write one of the vertices $\mathbf{v}(\mathbf{p})$ as a linear combination of these two vectors:

$$\mathbf{v}(\mathbf{p}) = \begin{bmatrix} \mathbf{f} & \mathbf{n} \end{bmatrix} \mathbf{a}(\mathbf{p}) = \begin{bmatrix} -n_2 & n_1 \\ n_1 & n_2 \end{bmatrix} \mathbf{a}(\mathbf{p}) \tag{5.40}$$

or

$$\mathbf{a}(\mathbf{p}) = \begin{bmatrix} -n_2 & n_1 \\ n_1 & n_2 \end{bmatrix}^{-1} \mathbf{v}(\mathbf{p}) = \begin{bmatrix} -n_2/d & n_1/d \\ n_1/d & n_2/d \end{bmatrix} \mathbf{v}(\mathbf{p}), \tag{5.41}$$

with $d = n_1^2 + n_2^2$. The lattice associated to the linear subspace orthogonal to the facet is the projection of Λ into this space. Since $\mathbf{n}$ is primitive, a basis for this lattice can be identified with $\mathbf{n}/d$. The coordinate of the whole facet in this space is therefore $da_2(\mathbf{p}) = \begin{bmatrix} n_1 & n_2 \end{bmatrix} \mathbf{v}(\mathbf{p})$, while the transverse cone is $da_2(\mathbf{p}) + \mathbb{R}_+$. Similarly, a linear functional $\boldsymbol{\xi}'$ projects onto a linear functional $\xi = \langle \boldsymbol{\xi}', \mathbf{n} \rangle / d$ in the linear subspace. Applying (5.37), with $y = \frac{n_1}{d} D_1 + \frac{n_2}{d} D_2$ and $t = \{-n_1 v_1(\mathbf{p}) - n_2 v_2(\mathbf{p})\}$, we therefore find

$$\begin{aligned} D_{P(\mathbf{p}),F(\mathbf{p})} &= -\sum_{n=0}^{\infty} \frac{b(n+1, \{-n_1 v_1(\mathbf{p}) - n_2 v_2(\mathbf{p})\})}{(n+1)!} \left(\frac{n_1}{d} D_1 + \frac{n_2}{d} D_2 \right)^n \\ &= -\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{b(i+j+1, \{-n_1 v_1(\mathbf{p}) - n_2 v_2(\mathbf{p})\})}{(i+j+1)!} \frac{n_1^i n_2^j}{d^{i+j}} D_1^i D_2^j. \end{aligned}$$

After applying this differential operator to the polynomial $h(\mathbf{p})(\mathbf{x})$, the resulting polynomial $h'(\mathbf{p})(\mathbf{x}) = D_{P(\mathbf{p}),F(\mathbf{p})} \cdot h(\mathbf{p})(\mathbf{x})$ needs to be integrated over the facet. The measure to be used is such that the integral of a lattice tile in the linear space parallel to the facet is 1, i.e.,

$$\int_{\mathbf{0}}^{\mathbf{f}} 1 = \int_0^1 1 dz = 1,$$

with z the coordinate along $\mathbf{f}$. Referring to (5.40) and (5.41), all points of the facet have the form $\mathbf{x}(\mathbf{p}) = z\,\mathbf{f} + a_2(\mathbf{p})\,\mathbf{n}$, while the z-coordinate of the vertices $\mathbf{v}_1(\mathbf{p})$ and $\mathbf{v}_2(\mathbf{p})$ are $(-n_2 v_{1,1} + n_1 v_{1,2})/d$ and $(-n_2 v_{2,1} + n_1 v_{2,2})/d$, respectively. That is, the contribution of the facet is equal to

$$\int_{(-n_2 v_{1,1}+n_1 v_{1,2})/d}^{(-n_2 v_{2,1}+n_1 v_{2,2})/d} h'(\mathbf{p})\,(z\,\mathbf{f} + a_2(\mathbf{p})\,\mathbf{n})\;dz,$$

where, again, we need to ensure that the lower limit is smaller than the upper limit using the usual method of plugging in a particular value of the parameters.

Finally, we consider the contributions of the vertices. The transverse cones are in this case simply the supporting cones. Since μ is a valuation, we may apply Barvinok's decomposition and assume that the cone is unimodular. For an affine cone

$$\begin{aligned} K &= \mathbf{v}(\mathbf{p}) + \mathbb{R}_+\mathbf{r}_1 + \mathbb{R}_+\mathbf{r}_2 \\ &= (a_1(\mathbf{p}) + \mathbb{R}_+)\mathbf{r}_1 + (a_2(\mathbf{p}) + \mathbb{R}_+)\mathbf{r}_2, \end{aligned}$$

with

$$\mathbf{a}(\mathbf{p}) = \begin{bmatrix}\mathbf{r}_1 & \mathbf{r}_2\end{bmatrix}^{-1} \mathbf{v}(\mathbf{p}),$$

we have (Berline and Vergne 2006, Proposition 31),

$$\mu(K)(\boldsymbol{\xi}) = \frac{e^{t_1 y_1 + t_2 y_2}}{(1-e^{y_1})(1-e^{y_2})} + \frac{1}{y_1} B(y_2 - C_1 y_1, t_2) + \frac{1}{y_2} B(y_1 - C_2 y_2, t_1) - \frac{1}{y_1 y_2}, \tag{5.42}$$

with

$$B(y,t) = \frac{e^{ty}}{1-e^y} + \frac{1}{y} = -\sum_{n=0}^{\infty} \frac{b(n+1,t)}{(n+1)!} y^n,$$

$y_i = \langle \boldsymbol{\xi}, \mathbf{r}_i \rangle$, $C_i = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle / \langle \mathbf{v}_i, \mathbf{v}_i \rangle$ and $t_i = \{-a_i(\mathbf{p})\}$. Expanding (5.42), we find

$$\begin{aligned} \mu(K)(\boldsymbol{\xi}) = &\left(-\frac{b(0,t1)}{y_1} - \sum_{n=0}^{\infty} \frac{b(n+1,t_1)}{(n+1)!} y_1^n\right)\left(-\frac{b(0,t2)}{y_2} - \sum_{n=0}^{\infty} \frac{b(n+1,t_2)}{(n+1)!} y_2^n\right) \\ &- \left(\sum_{n=0}^{\infty} \frac{b(n+1,t_2)}{(n+1)!} \frac{y_2^n}{y_1} + \sum_{n=0}^{\infty} \frac{b(n+1,t_2)}{(n+1)!} \frac{(y_2 - C_1 y_1)^n - y_2^n}{y_1}\right) \\ &- \left(\sum_{n=0}^{\infty} \frac{b(n+1,t_1)}{(n+1)!} \frac{y_1^n}{y_2} + \sum_{n=0}^{\infty} \frac{b(n+1,t_1)}{(n+1)!} \frac{(y_1 - C_2 y_2)^n - y_1^n}{y_2}\right) \\ &- \frac{1}{y_1 y_2} \\ = &\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} c(C_1, C_2, t_1, t_2; n_1, n_2)\, y_1^n y_2^n, \end{aligned}$$

with

$$\begin{aligned} c(C_1, C_2, t_1, t_2; n_1, n_2) = & \frac{b(n_1+1, t_1)}{(n_1+1)!} \frac{b(n_2+1, t_2)}{(n_2+1)!} \\ & - \frac{b(n_1+n_2+1, t_2)}{(n_1+n_2+1)!} \binom{n_1+n_2+1}{n_1+1} (-C_1)^{n_1+1} \\ & - \frac{b(n_1+n_2+1, t_1)}{(n_1+n_2+1)!} \binom{n_1+n_2+1}{n_2+1} (-C_2)^{n_2+1} . \end{aligned}$$

For $\boldsymbol{\xi} = \mathbf{D}$, we have

$$\begin{aligned} y_1^n y_2^n &= (r_{1,1} D_1 + r_{1,2} D_2)^{n_1} \, (r_{2,1} D_1 + r_{2,2} D_2)^{n_2} \\ &= \left(\sum_{k=0}^{n_1} r_{1,1}^k r_{1,2}^{n_1-k} \binom{n_1}{k} D_1^k D_2^{n_1-k} \right) \left(\sum_{l=0}^{n_2} r_{2,1}^l r_{2,2}^{n_2-l} \binom{n_2}{l} D_1^l D_2^{n_2-l} \right) \end{aligned}$$

and so $D_{P(\mathbf{p}),\mathbf{v}(\mathbf{p})} = \mu(K)(\mathbf{D}) =$

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\substack{i+j=n_1+n_2 \\ n_1 \geq 0 \\ n_2 \geq 0}} \sum_{\substack{k+l=i \\ 0 \leq k \leq n_1 \\ 0 \leq l \leq n_2}} c(C_1, C_2, t_1, t_2; n_1, n_2) r_{1,1}^k r_{1,2}^{n_1-k} r_{2,1}^l r_{2,2}^{n_2-l} \binom{n_1}{k} \binom{n_2}{l} D_1^i D_2^j .$$

The contribution of this vertex is then

$$h'(\mathbf{p})(\mathbf{v}(\mathbf{p})),$$

with $h'(\mathbf{p})(\mathbf{x}) = D_{P(\mathbf{p}),\mathbf{v}(\mathbf{p})} \cdot h(\mathbf{p})(\mathbf{x})$.

Example 5.43 *As a simple example, consider the (non-parametric) triangle in Figure 5.44 and assume we want to compute*

$$\sum_{\mathbf{x} \in T \cap \mathbb{Z}^2} x_1 x_2 .$$

Since $T \cap \mathbb{Z}^2 = \{ (2,4), (3,4), (2,5) \}$*, the result should be*

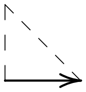
$$2 \cdot 4 + 3 \cdot 4 + 2 \cdot 5 = 30.$$

Let us first consider the integral

$$\iint_T x_1 x_2 = \int_{\partial T} \frac{x_1^2 x_2}{2} \, dx_2 .$$

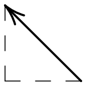
Integration along each of the edges of the triangle yields the following.

For the edge in the margin, we have $\mathbf{f} = (1, 0)$*, i.e.,* $f_2 = 0$*. The contribution of this edge to the integral is therefore zero.*



For this edge, we have $\mathbf{f} = (-1, 1)$*. The contribution of this edge to the integral is therefore*

$$\int_0^1 \frac{(3-\lambda)^2 (4+\lambda)}{2} d\lambda = \frac{337}{24} .$$



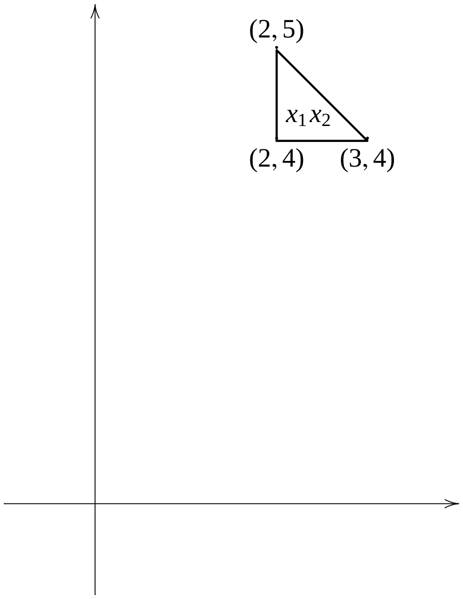
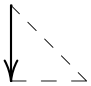


Figure 5.44: Sum of polynomial x_1x_2 over the integer points in a triangle T

For this edge, we have $\mathbf{f} = (0, -1)$. *The contribution of this edge to the integral is therefore*



$$\int_0^1 \frac{2^2(5-\lambda)}{2}(-1)d\lambda = -9.$$

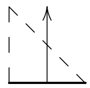
The total integral is therefore

$$\int_{\partial T} \frac{x_1^2 x_2}{2}\,dx_2 = 0 + \frac{337}{24} - 9 = \frac{121}{24}.$$

Now let us consider the contributions of the edges. We will need the following Bernoulli numbers in our computations.

$$\begin{aligned} b(1,0) &= -\frac{1}{2} \\ b(2,0) &= \frac{1}{6} \\ b(3,0) &= 0 \\ b(4,0) &= -\frac{1}{30} \end{aligned}$$

The normal to the facet F_1 *in the margin is* $\mathbf{n} = (0, 1)$. *The vector* $\mathbf{f} = (-1, 0)$ *is parallel to the facet. We have*



$$\begin{bmatrix}2\\4\end{bmatrix} = -2\begin{bmatrix}-1\\0\end{bmatrix} + 4\begin{bmatrix}0\\1\end{bmatrix} \quad \textit{and} \quad \begin{bmatrix}3\\4\end{bmatrix} = -3\begin{bmatrix}-1\\0\end{bmatrix} + 4\begin{bmatrix}0\\1\end{bmatrix}.$$

Therefore $t = \{-4\} = 0$, $y = D_2$,

$$\begin{aligned} D_{T,F_1} &= -\sum_{j=0}^{\infty} \frac{b(j+1,0)}{(j+1)!} D_2^j \\ &= -\frac{b(1,0)}{1} - \frac{b(2,0)}{2} D_2 + \cdots \end{aligned}$$

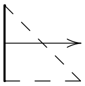
and

$$h'(\mathbf{x}) = D_{T,F_1} \cdot x_1 x_2 = \left(\frac{1}{2} - \frac{1}{12} D_2\right) \cdot x_1 x_2 = \frac{1}{2} x_1 x_2 - \frac{1}{12} x_1.$$

With $x_1 = -z$ *and* $x_2 = 4$, *the contribution of this facet is*

$$\int_{-3}^{-2} -2z + \frac{1}{12} z \, dz = \frac{115}{24}.$$

The normal to the facet F_2 *in the margin is* $\mathbf{n} = (1, 0)$. *The vector* $\mathbf{f} = (0, 1)$ *is parallel to the facet. We have*



$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} 2 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore $t = \{-2\} = 0$, $y = D_1$,

$$\begin{aligned} D_{T,F_2} &= -\sum_{i=0}^{\infty} \frac{b(i+1,0)}{(i+1)!} D_1^i \\ &= -\frac{b(1,0)}{1} - \frac{b(2,0)}{2} D_1 + \cdots \end{aligned}$$

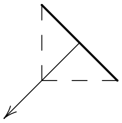
and

$$h'(\mathbf{x}) = D_{T,F_2} \cdot x_1 x_2 = \left(\frac{1}{2} - \frac{1}{12} D_1\right) \cdot x_1 x_2 = \frac{1}{2} x_1 x_2 - \frac{1}{12} x_2.$$

With $x_1 = 2$ *and* $x_2 = z$, *the contribution of this facet is*

$$\int_{4}^{5} z - \frac{1}{12} z \, dz = \frac{33}{8}.$$

The normal to the facet F_3 *in the margin is* $\mathbf{n} = (-1, -1)$. *The vector* $\mathbf{f} = (1, -1)$ *is parallel to the facet. We have*



$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \frac{7}{2} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} 2 \\ 5 \end{bmatrix} = -\frac{3}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \frac{7}{2} \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Therefore $t = \{7\} = 0$, $y = -\frac{1}{2} D_1 - \frac{1}{2} D_2$,

$$\begin{aligned} D_{T,F_3} &= -\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{b(i+j+1,0)}{(i+j+1)!} \frac{(-1)^{i+j}}{2^{i+j}} D_1^i D_2^j \\ &= -\frac{b(1,0)}{1} + \frac{1}{2} \frac{b(2,0)}{2} D_1 + \frac{1}{2} \frac{b(2,0)}{2} D_2 + \cdots \end{aligned}$$

and

$$h'(\mathbf{x}) = D_{T,F_4} \cdot x_1x_2 = \left(\frac{1}{2} + \frac{1}{24}D_1 + \frac{1}{24}D_2\right) \cdot x_1x_2 = \frac{1}{2}x_1x_2 + \frac{1}{24}x_2 + \frac{1}{24}x_1.$$

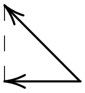
With $x_1 = z + \frac{7}{2}$ *and* $x_2 = -z + \frac{7}{2}$*, the contribution of this facet is*

$$\int_{-\frac{3}{2}}^{-\frac{1}{2}} \frac{1}{2}(z + \frac{7}{2})(-z + \frac{7}{2}) + \frac{1}{24}(-z + \frac{7}{2}) + \frac{1}{24}(z + \frac{7}{2})\, dz = \frac{47}{8}.$$

The total contribution of the edges is therefore

$$\frac{115}{24} + \frac{33}{8} + \frac{47}{8} = \frac{355}{24}.$$

Finally, we consider the contributions of the vertices.



For the vertex $\mathbf{v} = (3,4)$*, we have* $\mathbf{r}_1 = (-1,0)$ *and* $\mathbf{r}_2 = (-1,1)$*. Since* $\mathbf{v}$ *is integer, we have* $t_1 = t_2 = 0$*. Also,* $C_1 = 1$*,* $C_2 = 1/2$*,* $y_1 = -D_1$ *and* $y_2 = -D_1 + D_2$*. Since the total degree of the polynomial* x_1x_2 *is two, we only need the coefficients of* $\mu(K)(\boldsymbol{\xi})$ *up to* $n_1 + n_2 = 2$.

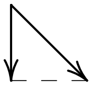
n_1	n_2	
0	0	$\left(\frac{b(1,0)}{1!}\frac{b(1,0)}{1!} - \frac{b(2,0)}{2!}\binom{1}{1}(-1)^1 - \frac{b(2,0)}{2!}\binom{1}{1}(-\frac{1}{2})^1\right)$
1	0	$\left(\frac{b(2,0)}{2!}\frac{b(1,0)}{1!} - \frac{b(3,0)}{3!}\binom{2}{2}(-1)^2 - \frac{b(3,0)}{3!}\binom{2}{1}(-\frac{1}{2})^1\right)(-D_1)$
0	1	$\left(\frac{b(1,0)}{1!}\frac{b(2,0)}{2!} - \frac{b(3,0)}{3!}\binom{2}{1}(-1)^1 - \frac{b(3,0)}{3!}\binom{2}{2}(-\frac{1}{2})^2\right)(-D_1 + D_2)$
2	0	$\left(\frac{b(3,0)}{3!}\frac{b(1,0)}{1!} - \frac{b(4,0)}{4!}\binom{3}{3}(-1)^3 - \frac{b(4,0)}{4!}\binom{3}{1}(-\frac{1}{2})^1\right)(-D_1)^2$
1	1	$\left(\frac{b(2,0)}{2!}\frac{b(2,0)}{2!} - \frac{b(4,0)}{4!}\binom{3}{2}(-1)^2 - \frac{b(4,0)}{4!}\binom{3}{2}(-\frac{1}{2})^2\right)(-D_1)(-D_1 + D_2)$
0	2	$\left(\frac{b(1,0)}{1!}\frac{b(3,0)}{3!} - \frac{b(4,0)}{4!}\binom{3}{1}(-1)^1 - \frac{b(4,0)}{4!}\binom{3}{3}(-\frac{1}{2})^3\right)(-D_1 + D_2)^2$

We find

$$\begin{aligned} h'(\mathbf{x}) &= \left(\frac{3}{8} - \frac{1}{24}(-D_1) - \frac{1}{24}(-D_1 + D_2) + \frac{7}{576}(-D_1D_2) - \frac{5}{1152}(-2D_1D2)\right)x_1x_2 \\ &= \frac{3}{8}x_1x_2 + \frac{1}{24}x_2 - \frac{1}{24}(-x_2 + x_1) + \frac{7}{576}(-1) - \frac{5}{1152}(-2). \end{aligned}$$

The contribution of this vertex is therefore

$$h'(3,4) = \frac{1355}{288}.$$



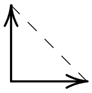
For the vertex $\mathbf{v} = (2,5)$*, we have* $\mathbf{r}_1 = (0,-1)$ *and* $\mathbf{r}_2 = (1,-1)$*. Since* $\mathbf{v}$ *is integer, we have* $t_1 = t_2 = 0$*. Also,* $C_1 = 1$*,* $C_2 = 1/2$*,* $y_1 = -D_2$ *and* $y_2 = D_1 - D_2$*. We similarly find*

$$h'(\mathbf{x}) = \frac{3}{8}x_1x_2 + \frac{1}{24}x_1 - \frac{1}{24}(x_2 - x_1) + \frac{7}{576}(-1) - \frac{5}{1152}(-2).$$

The contribution of this vertex is therefore

$$h'(2,5) = \frac{1067}{288}.$$

For the vertex $\mathbf{v} = (2, 4)$, *we have* $\mathbf{r}_1 = (1, 0)$ *and* $\mathbf{r}_2 = (0, 1)$. *Since* $\mathbf{v}$ *is integer, we have* $t_1 = t_2 = 0$. *The computations are easier in this case since* $C_1 = C_2 = 0$, $y_1 = D_1$ *and* $y_2 = D_2$. *We find*



$$h'(\mathbf{x}) = \frac{1}{4}x_1x_2 - \frac{1}{12}x_2 - \frac{1}{12}x_1 + \frac{1}{144}(1).$$

The contribution of this vertex is therefore

$$h'(2, 4) = \frac{253}{144}.$$

The total contribution of the vertices is then

$$\frac{1355}{288} + \frac{1067}{288} + \frac{253}{144} = \frac{61}{6}$$

and the total sum is

$$\frac{121}{24} + \frac{355}{24} + \frac{61}{6} = 30.$$

Example 5.45 *Consider the parametric polytope*

$$P(n) = \{ \mathbf{x} \mid x_1 \geq 2 \wedge 3x_1 \leq n + 9 \wedge 4 \leq x_2 \leq 5 \}.$$

If $n \geq -3$, *then the vertices of this polytope are* $(2, 4)$, $(2, 5)$, $(3+n/3, 4)$ *and* $(3+n/3, 5)$. *The contributions of the faces of* $P(n)$ *to*

$$\sum_{\mathbf{x} \in P(n) \cap \mathbb{Z}^2} x_1x_2$$

for the chamber $n \geq -3$ *are shown in Table 2. The final result is*

$$\begin{cases} \frac{n^2}{2} - 3n\left\{\frac{n}{3}\right\} + \frac{21}{2}n + \frac{9}{2}\left\{\frac{n}{3}\right\}^2 - \frac{63}{2}\left\{\frac{n}{3}\right\} + 45 & \text{if } n + 3 \geq 0. \end{cases}$$

5.15 Summation through exponential substitution and Laurent expansions

This section was inspired by Baldoni et al. (2008).

Let $f(\mathbf{x})$ be the generating function of a polytope P, i.e.,

$$f(\mathbf{x}) = \sum_{\mathbf{t} \in P \cap \mathbb{Z}^d} \mathbf{x}^{\mathbf{t}}.$$

Substituting $\mathbf{x} = \mathbf{e}^{\mathbf{y}}$, we obtain

$$f(\mathbf{e}^{\mathbf{y}}) = \sum_{\mathbf{t} \in P \cap \mathbb{Z}^d} e^{\langle \mathbf{t}, \mathbf{y} \rangle} = \sum_{\mathbf{t} \in P \cap \mathbb{Z}^d} \sum_{\mathbf{n} \geq \mathbf{0}} \frac{\mathbf{t}^{\mathbf{n}} \mathbf{y}^{\mathbf{n}}}{\mathbf{n}!} = \sum_{\mathbf{n} \geq \mathbf{0}} \left(\sum_{\mathbf{t} \in P \cap \mathbb{Z}^d} \mathbf{t}^{\mathbf{n}} \right) \frac{\mathbf{y}^{\mathbf{n}}}{\mathbf{n}!},$$

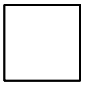
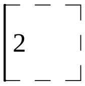
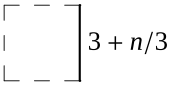
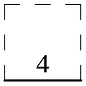
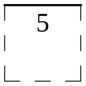
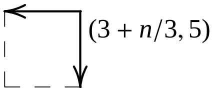
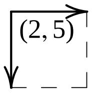
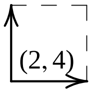
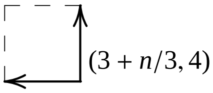
	$\frac{n^2}{4} + \frac{9}{2}n + \frac{45}{4}$
	$\frac{33}{8}$
	$-\frac{3}{2}n\left\{\frac{n}{3}\right\} + \frac{3}{4}n + \frac{9}{4}\left\{\frac{n}{3}\right\}^2 - \frac{63}{4}\left\{\frac{n}{3}\right\} + \frac{57}{8}$
	$\frac{23}{216}n^2 + \frac{23}{12}n + \frac{115}{24}$
	$\frac{31}{216}n^2 + \frac{31}{12}n + \frac{155}{24}$
	$-\frac{31}{36}n\left\{\frac{n}{3}\right\} + \frac{31}{72}n + \frac{31}{24}\left\{\frac{n}{3}\right\}^2 - \frac{217}{24}\left\{\frac{n}{3}\right\} + \frac{589}{144}$
	$\frac{341}{144}$
	$\frac{253}{144}$
	$-\frac{23}{36}n\left\{\frac{n}{3}\right\} + \frac{23}{72}n + \frac{23}{24}\left\{\frac{n}{3}\right\}^2 - \frac{161}{24}\left\{\frac{n}{3}\right\} + \frac{437}{144}$

Table 2: Contributions of the faces of $P(n)$ to the sum of x_1x_2 over the integer points of $P(n)$

with $\mathbf{n}! = n_1!n_2!\cdots n_d!$. We observe that the sum of the monomial $\mathbf{t}^{\mathbf{n}}$ over the integer points in P is equal to $\mathbf{n}!$ times the coefficient of the $\mathbf{y}^{\mathbf{n}}$ term in the Taylor expansion of $f(\mathbf{e}^{\mathbf{y}})$.

As in the case of unweighted counting (see subsection 5.11), we have to add the coefficients of these monomials in the Laurent expansions of the terms in (5.26). However, unlike the case of unweighted counting, we cannot transform this problem to a univariate problem and computing the coefficient of a monomial in the Laurent expansions does not reduce to computing the coefficient of a single higher-degree monomial in a Taylor expansion.

Consider now one of the terms $g(\mathbf{x}) = f_{ik}(\mathbf{x})$ in (5.26),

$$g(\mathbf{e}^{\mathbf{y}}) = \frac{e^{\sum_{j=1}^d s_j(\mathbf{p})\langle \mathbf{b}_j, \mathbf{y}\rangle}}{\prod_{j=1}^d \left(1 - e^{\langle \mathbf{b}_j, \mathbf{y}\rangle}\right)},$$

with $\mathbf{w}_{ij}(\mathbf{p}) = \sum_{j=1}^d s_j(\mathbf{p})\mathbf{b}_j$ written in terms of the $\mathbf{b}_j$, which are assumed to form a basis and where we have made explicit the only place where the parameters $\mathbf{p}$ appear. We rewrite this equation as

$$g(\mathbf{e}^{\mathbf{y}}) = \left(\prod_{j=1}^d \frac{-1}{\langle \mathbf{b}_j, \mathbf{y}\rangle}\right)\left(\prod_{j=1}^d \frac{-\langle \mathbf{b}_j, \mathbf{y}\rangle\, e^{s_j(\mathbf{p})\langle \mathbf{b}_j, \mathbf{y}\rangle}}{1 - e^{\langle \mathbf{b}_j, \mathbf{y}\rangle}}\right). \tag{5.46}$$

The second factor is analytic and is a product of generating functions $\mathrm{Todd}(s_j(\mathbf{p}), \langle \mathbf{b}_j, \mathbf{y}\rangle)$ of Bernoulli polynomials (5.38). Plugging in these expressions, we find

$$\begin{aligned}
\mathrm{Todd}(s_j(\mathbf{p}), \langle \mathbf{b}_j, \mathbf{y}\rangle) &= \frac{-\langle \mathbf{b}_j, \mathbf{y}\rangle e^{s_j(\mathbf{p})\langle \mathbf{b}_j, \mathbf{y}\rangle}}{1 - e^{\langle \mathbf{b}_j, \mathbf{y}\rangle}} \\
&= \sum_{n=0}^{\infty} \frac{b(n, s_j(\mathbf{p}))}{n!} \langle \mathbf{b}_j, \mathbf{y}\rangle^n \\
&= \sum_{\mathbf{k} \geq \mathbf{0}} \frac{b(\sum k_i, s_j(\mathbf{p}))}{(\sum k_i)!} \binom{\sum k_i}{\mathbf{k}} \mathbf{b}_j^{\mathbf{k}} \mathbf{y}^{\mathbf{k}} \\
&= \sum_{\mathbf{k} \geq \mathbf{0}} \frac{b(\sum k_i, s_j(\mathbf{p}))}{\prod_i k_i!} \mathbf{b}_j^{\mathbf{k}} \mathbf{y}^{\mathbf{k}},
\end{aligned} \tag{5.47}$$

with

$$\binom{\sum k_i}{\mathbf{k}} = \binom{\sum k_i}{k_1, k_2, \ldots k_d} = \frac{(\sum k_i)!}{\mathbf{k}!} = \prod_{i=1}^d \binom{\sum_{j=1}^i k_j}{k_i}$$

the multinomial coefficients. For the first factor, we compute the Laurent expansion of each of its factors,

$$\begin{aligned}
\frac{-1}{\langle \mathbf{b}_j, \mathbf{y}\rangle} &= \frac{-1}{\sum_{k=f}^{d} b_{jk} y_k} \\
&= \frac{-1}{b_{jf} y_f \left(1 + \frac{\sum_{k=f+1}^{d} b_{jk} y_k}{b_{jf} y_f}\right)} \\
&= \frac{-1}{b_{jf} y_f} \sum_{n=0}^{\infty} (-1)^n \left(\frac{\sum_{k=f+1}^{d} b_{jk} y_k}{b_{jf} y_f}\right)^n \\
&= \sum_{\mathbf{n} \geq \mathbf{0}} \binom{\sum n_k}{\mathbf{n}} (-1)^{\sum n_k + 1} \frac{\mathbf{b}_j'^{\mathbf{n}}}{b_{jf}^{\sum n_k + 1}} \frac{\mathbf{y}'^{\mathbf{n}}}{y_f^{\sum n_k + 1}},
\end{aligned} \tag{5.48}$$

where b_{jf} is the first non-zero coefficient of $\mathbf{b}_j$ and the vector $\mathbf{b}_j'$ contains the subsequent $d - f$ coefficients of $\mathbf{b}_j$.

Given a polynomial

$$q(\mathbf{y}, \mathbf{p}) = \sum_{\mathbf{m}} \beta_{\mathbf{m}}(\mathbf{p})\, \mathbf{y}^{\mathbf{m}}$$

that we wish to sum over the integer points of a polytope P, we perform the following operations for each unimodular cone in the decomposition of each vertex cone.

- For each $\mathbf{m}$ with $\beta_{\mathbf{m}}(\mathbf{p}) \neq 0$
  - Compute all sums $\mathbf{N} = \sum_{j=1}^{d} (\mathbf{0}, -\sum_k n_{jk} - 1, \mathbf{n}_j)$ of exponents from (5.48) such that $\mathbf{N} \leq \mathbf{m}$ and compute the corresponding coefficient $\gamma_{\mathbf{N}}$ in the product of Laurent series by enumerating all combinations of $\mathbf{n}_j$ leading to the same $\mathbf{N}$. Note that there are only a finite number of $\mathbf{N}$ satisfying this constraint since $\sum N_k = -d$. By reordering the variables such that the highest exponents occurs for the first variable, the number of $\mathbf{N}$ can be reduced.
  - For each of these $\mathbf{N}$
    - Compute the coefficient $\delta_{\mathbf{m}-\mathbf{N}}(\mathbf{p})$ of $\mathbf{y}^{\mathbf{m}-\mathbf{N}}$ in the product of Taylor expansions (5.47).
- The contribution of this cone is the sum of

  $$\mathbf{m}!\, \alpha\, \beta_{\mathbf{m}}(\mathbf{p})\, \gamma_{\mathbf{N}}\, \delta_{\mathbf{m}-\mathbf{N}}(\mathbf{p})$$

  over all considered $\mathbf{m}$ and $\mathbf{N}$.

Within each vertex cone computation, the coefficients $\gamma_{\mathbf{N}}$ and $\delta_{\mathbf{m}-\mathbf{N}}(\mathbf{p})$ only need to be computed once.

Example 5.49 *Consider once more the rational generating function*

$$f(T; \mathbf{x}) = \frac{x_1^2}{(1 - x_1^{-1})(1 - x_1^{-1} x_2)} + \frac{x_2^2}{(1 - x_2^{-1})(1 - x_1 x_2^{-1})} + \frac{1}{(1 - x_1)(1 - x_2)}$$

from Verdoolaege (2005, Example 39) and Example 5.30. Assume we want to compute

$$\sum_{\mathbf{y} \in T \cap \mathbb{Z}^2} y_1^2 + y_2^2.$$

We will need the following Bernoulli polynomials

$$\begin{aligned}
b(0, s) &= 1 \\
b(1, s) &= \frac{1}{2}(-1 + 2s) \\
b(2, s) &= \frac{1}{6}\left(1 - 6s + 6s^2\right) \\
b(3, s) &= \frac{1}{2}\left(s - 3s^2 + 2s^3\right) \\
b(4, s) &= \frac{1}{30}\left(-1 + 30s^2 - 60s^3 + 30s^4\right)
\end{aligned}$$

For the first term, substitution yields

$$\begin{aligned}
h(\mathbf{y}) =& \frac{1}{y_1} \frac{1}{y_1 - y_2} \frac{y_1 e^{(-2)(-y_1)}}{1 - e^{-y_1}} \frac{(y_1 - y2) e^{0(-y_1 + y_2)}}{1 - e^{-y_1 + y_2}} \\
=& \frac{1}{y_1} \left(\frac{1}{y_1} \left(1 + \frac{y_2}{y_1} + \frac{y_2^2}{y_1^2} + \cdots \right) \right) \\
& \left(1 + \frac{b(1, -2)}{1}(-y_1) + \frac{b(2, -2)}{2}(-y_1)^2 + \frac{b(3, -2)}{3!}(-y_1)^3 + \frac{b(4, -2)}{4!}(-y_1)^4 + \cdots \right) \\
& \left(1 + \frac{-1}{2}(-y_1 + y_2) + \frac{1}{12}(-y_1 + y_2)^2 + 0(-y_1 + y_2)^3 + \frac{1}{720}(-y_1 + y_2)^4 + \cdots \right)
\end{aligned}$$

We obtain the following results:

$\mathbf{m}$	$\mathbf{N}$	$\gamma_{\mathbf{N}} \mathbf{y}^{\mathbf{N}}$	$\mathbf{m} - \mathbf{N}$	$\delta_{\mathbf{m}-\mathbf{N}} \mathbf{y}^{\mathbf{m}-\mathbf{N}}$	$\mathbf{m}! \alpha \beta_{\mathbf{m}} \gamma_{\mathbf{N}} \delta_{\mathbf{m}-\mathbf{N}}$
$(2, 0)$	$(-2, 0)$	$1 y_1^{-2}$	$(4, 0)$	$\frac{721}{240} y_1^4$	$\frac{721}{120}$
$(0, 2)$	$(-2, 0)$	$1 y_1^{-2}$	$(2, 2)$	$\frac{179}{720} y_1^2 y_2^2$	$\frac{179}{360}$
	$(-3, 1)$	$1 y_1^{-3} y_2$	$(3, 1)$	$-\frac{211}{120} y_1^3 y_1$	$-\frac{211}{60}$
	$(-4, 2)$	$1 y_1^{-4} y_2^2$	$(4, 0)$	$\frac{721}{240} y_1^4$	$\frac{721}{120}$

For the second term, we similarly obtain

$\mathbf{m}$	$\mathbf{N}$	$\gamma_{\mathbf{N}}\mathbf{y}^{\mathbf{N}}$	$\mathbf{m}-\mathbf{N}$	$\delta_{\mathbf{m}-\mathbf{N}}\mathbf{y}^{\mathbf{m}-\mathbf{N}}$	$\mathbf{m}!\alpha\beta_{\mathbf{m}}\gamma_{\mathbf{N}}\delta_{\mathbf{m}-\mathbf{N}}$
$(2,0)$	$(-1,-1)$	$-1y_1^{-1}y_2^{-1}$	$(3,1)$	$\frac{1}{180}y_1^3y_1$	$-\frac{1}{90}$
	$(-2,0)$	$-1y_1^{-2}$	$(4,0)$	$-\frac{1}{720}y_1^4$	$\frac{1}{360}$
$(0,2)$	$(-1,-1)$	$-1y_1^{-1}y_2^{-1}$	$(1,3)$	$-\frac{211}{120}y_1y_2^3$	$\frac{211}{60}$
	$(-2,0)$	$-1y_1^{-3}y_2$	$(2,2)$	$\frac{179}{720}y_1^3y_2$	$-\frac{179}{360}$
	$(-3,1)$	$-1y_1^{-3}y_2$	$(3,1)$	$\frac{1}{180}y_1^3y_1$	$-\frac{1}{90}$
	$(-4,2)$	$-1y_1^{-4}y_2^2$	$(4,0)$	$-\frac{1}{720}y_1^4$	$\frac{1}{360}$

Finally, for the third term, we obtain

$\mathbf{m}$	$\mathbf{N}$	$\gamma_{\mathbf{N}}\mathbf{y}^{\mathbf{N}}$	$\mathbf{m}-\mathbf{N}$	$\delta_{\mathbf{m}-\mathbf{N}}\mathbf{y}^{\mathbf{m}-\mathbf{N}}$	$\mathbf{m}!\alpha\beta_{\mathbf{m}}\gamma_{\mathbf{N}}\delta_{\mathbf{m}-\mathbf{N}}$
$(2,0)$	$(-1,-1)$	$-1y_1^{-1}y_2^{-1}$	$(3,1)$	$0y_1^3y_1$	0
$(0,2)$	$(-1,-1)$	$-1y_1^{-1}y_2^{-1}$	$(1,3)$	$0y_1y_2^3$	0

Adding up all contributions in the final columns of these tables, we obtain a grand total of

$$12.$$

5.16 Conversion to "standard form"

Some algorithms or tools expect a polyhedron to be specified in "standard form", i.e.,

$$\begin{cases} A\mathbf{x} = \mathbf{b} \\ \mathbf{x} \ge \mathbf{0}. \end{cases} \tag{5.50}$$

Given an arbitrary (parametric) polyhedron

$$\{\, \mathbf{x} \mid A\mathbf{x} + \mathbf{b}(\mathbf{p}) \ge 0 \,\}, \tag{5.51}$$

a conversion to standard form requires the introduction of slack variables and a way of dealing with variables of unrestricted sign. In this section we will be satisfied with a reduction to the form

$$\begin{cases} A\mathbf{x} = \mathbf{b} \\ D\mathbf{x} \ge \mathbf{c}, \end{cases} \tag{5.52}$$

with D a diagonal matrix with positive entries. That is, we do not necessarily make all variables non-negative, but we do ensure that they have a lower bound. If needed, a subsequent reduction can then be performed.

The standard way of dealing with variables of unrestricted sign is to replace a variable x of unknown sign by the difference ($x = x' - x''$) of two non-negative variables

($x', x'' \geq 0$). However, some algorithms are somewhat sensitive with respect to the number of variables and so we would prefer to introduce as few extra variables as possible. We will therefore apply a unimodular transformation on the variables such that all transformed variables are known to be non-negative.

The first step is to compute the HNF of A, i.e., a matrix $H = AU$, with U unimodular, in column echelon form such that the first entry in each column is positive and the other entries on the corresponding row are non-negative and strictly smaller than this first entry. By reordering the rows we may assume that the top square part of H is lower-triangular. By a further unimodular transformation, the entries below the diagonal can be made non-positive and strictly smaller (in absolute value) than the diagonal entry of the same row.

For each of the new variables, we can take a positive combination of the corresponding row and the previous rows to obtain a positive multiple of the corresponding unit vector, implying that the variable has a lower bound. A slack variable can then be introduced for each of the rows in the top square part of H' that is not already a positive multiple of a unit vector and for each of the rows below the top square part of H'.

Example 5.53 *Consider the cone*

$$\left\{ \mathbf{x} \mid \begin{bmatrix} 67 & -13 \\ -52 & 53 \end{bmatrix} \mathbf{x} \geq \mathbf{0} \right\}.$$

This cone is already situated in the first quadrant, but this may not be obvious from the constraints. Furthermore, directly adding slack variables would lead to a total of 4 variables, whereas we can also represent this cone in standard form with only 3 variables. We have

$$H' = \begin{bmatrix} 1 & 0 \\ -1331 & 2875 \end{bmatrix} = \begin{bmatrix} 67 & -13 \\ -52 & 53 \end{bmatrix} \begin{bmatrix} -6 & 13 \\ -31 & 57 \end{bmatrix} = AU'.$$

Adding a slack variable for the second row of H', we obtain the equivalent problem

$$\left\{ \begin{aligned} \begin{bmatrix} -1331 & 2875 & -1 \end{bmatrix} \mathbf{x}' &= \mathbf{0} \\ \mathbf{x}' &\geq \mathbf{0} \end{aligned} \right.$$

with

$$\mathbf{x} = \begin{bmatrix} -6 & 13 & 0 \\ -31 & 57 & 0 \end{bmatrix} \mathbf{x}'.$$

A similar construction was used by Eisenbrand (2000, Lemma 3.10) and Hung and Rom (1990).

5.17 Using TOPCOM to compute Chamber Decompositions

In this section, we describe how to use the correspondence between the regular triangulations of a point set and the chambers of the Gale transform of the point set (Gelfand et al. 1994) to compute the chamber decomposition of a parametric polytope. This correspondence was also used by Pfeifle and Rambau (2003) Eisenschmidt and Köppe (2007).

Let us first assume that the parametric polytope can be written as

$$\begin{cases} \mathbf{x} \geq 0 \\ A\,\mathbf{x} \leq \mathbf{b}(\mathbf{p}), \end{cases} \tag{5.54}$$

where the right hand side $\mathbf{b}(\mathbf{p})$ is arbitrary and may depend on the parameters. The first step is to add slack variables $\mathbf{s}$ to obtain the vector partition problem

$$\begin{cases} A\,\mathbf{x} + I\,\mathbf{s} = \mathbf{b}(\mathbf{p}) \\ \qquad \mathbf{x}, \mathbf{s} \geq 0, \end{cases}$$

with I the identity matrix. Then we compute the (right) kernel K of the matrix $\begin{bmatrix} A & I \end{bmatrix}$, i.e.,

$$\begin{bmatrix} A & I \end{bmatrix} K = 0$$

and use `TOPCOM`'s `points2triangs` to compute the regular triangulations of the points specified by the rows of K. Each of the resulting triangulations corresponds to a chamber in the chamber complex of the above vector partition problem. Each simplex in a triangulation corresponds to a parametric vertex active on the corresponding chamber and each point in the simplex (i.e., a row of K) corresponds to a variable (x_j or s_j) that is set to zero to obtain this parametric vertex. In the original formulation of the problem (5.54) each such variable set to zero reflects the saturation of the corresponding constraint ($x_j = 0$ for $x_j = 0$ and $\langle \mathbf{a}_j, \mathbf{x} \rangle = b_j(\mathbf{p})$ for $s_j = 0$). A description of the chamber can then be obtained by plugging in the parametric vertices in the remaining constraints.

Example 5.55 *Consider the parametric polytope*

$$P(p,q,r) = \{\, (i,j) \mid 0 \leq i \leq p \wedge 0 \leq j \leq 2i + q \wedge 0 \leq k \leq i - p + r \wedge p \geq 0 \wedge q \geq 0 \wedge r \geq 0 \,\}.$$

The constraints involving the variables are

$$\begin{cases} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix} \begin{matrix} \geq 0 \\ \geq 0 \\ \geq 0 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix} \begin{matrix} \leq p \\ \leq q \\ \leq -p + r \end{matrix} \end{cases}$$

We have

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ -2 & 0 & -1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

Computing the regular triangulations of the rows of K using `TOPCOM`*, we obtain*

```
> cat e2.topcom
[
[  -1    0    0 ]
[  -2    0   -1 ]
[  -1   -1    0 ]
[   1    0    0 ]
[   0    1    0 ]
[   0    0    1 ]
]
> points2triangs --regular < e2.topcom
T[1]:={{0,1,2},{1,2,3},{0,1,4},{1,3,4},{0,2,5},{2,3,5},{0,4,5},{3,4,5}};
T[2]:={{1,2,3},{1,3,4},{2,3,5},{3,4,5},{1,2,5},{1,4,5}};
T[3]:={{1,2,3},{1,3,4},{2,3,5},{3,4,5},{1,2,4},{2,4,5}};
```

We see that we have three chambers in the decomposition, one with 8 vertices and two with 6 vertices. Take the second vertex (“`{1,2,3}`”) of the first chamber. This vertex corresponds to the saturation of the constraints $j \geq 0$, $k \geq 0$ *and* $i \leq p$, *i.e.,* $(i, j, k) = (p, 0, 0)$. *Plugging in this vertex in the remaining constraints, we see that it is only valid in case* $p \geq 0$, $r \geq 0$ *and* $2p + q \geq 0$. *For the remaining vertices of the first chamber, we similarly find*

`{0,1,2}`	$(0, 0, 0)$	$p \geq 0$, $-q + r \geq 0$ *and* $q \geq 0$
`{1,2,3}`	$(p, 0, 0)$	$p \geq 0$, $r \geq 0$ *and* $2p + q \geq 0$
`{0,1,4}`	$(0, 0, -p + r)$	$-q + r \geq 0$, $p \geq 0$ *and* $q \geq 0$
`{1,3,4}`	$(p, 0, r)$	$p \geq 0$, $r \geq 0$ *and* $2p + q \geq 0$
`{0,2,5}`	$(0, q, 0)$	$q \geq 0$, $p \geq 0$ *and* $-q + r \geq 0$
`{2,3,5}`	$(p, 2p + q, 0)$	$p \geq 0$, $2p + q \geq 0$ *and* $r \geq 0$
`{0,4,5}`	$(0, q, -p + r)$	$q \geq 0$, $-q + r \geq 0$ *and* $p \geq 0$
`{3,4,5}`	$(p, 2p + q, r)$	$p \geq 0$, $2p + q \geq 0$ *and* $r \geq 0$

Combining these constraints with the initial constraints of the problem on the parameters $p \geq 0$, $q \geq 0$ *and* $r \geq 0$, *we find the chamber* $\{\, (p, q, r) \mid p \geq 0 \wedge -p + r \geq 0 \wedge q \geq 0 \,\}$. *For the second chamber, we have*

`{1,2,3}`	$(p, 0, 0)$	$p \geq 0$, $r \geq 0$ *and* $2p + q \geq 0$
`{1,3,4}`	$(p, 0, r)$	$p \geq 0$, $r \geq 0$ *and* $2p + q \geq 0$
`{2,3,5}`	$(p, 2p + q, 0)$	$p \geq 0$, $2p + q \geq 0$ *and* $r \geq 0$
`{3,4,5}`	$(p, 2p + q, r)$	$p \geq 0$, $2p + q \geq 0$ *and* $r \geq 0$
`{1,2,5}`	$(-\frac{q}{2}, 0, 0)$	$-q \geq 0$, $2p + q \geq 0$ *and* $-2p - q + 2r \geq 0$
`{1,4,5}`	$(-\frac{q}{2}, 0, -p - \frac{q}{2} + r)$	$-q \geq 0$, $-2p - q + 2r \geq 0$ *and* $2p + q \geq 0$

The chamber is therefore $\{\, (p, q, r) \mid q = 0 \wedge p \geq 0 \wedge -p + r \geq 0 \,\}$. *Note that by intersecting with the initial constraints this chamber is no longer full-dimensional and can therefore be discarded. Finally, for the third chamber, we have*

`{1,2,3}`	$(p, 0, 0)$	$p \geq 0$, $r \geq 0$ *and* $2p + q \geq 0$
`{1,3,4}`	$(p, 0, r)$	$p \geq 0$, $r \geq 0$ *and* $2p + q \geq 0$
`{2,3,5}`	$(p, 2p + q, 0)$	$p \geq 0$, $2p + q \geq 0$ *and* $r \geq 0$
`{3,4,5}`	$(p, 2p + q, r)$	$p \geq 0$, $2p + q \geq 0$ *and* $r \geq 0$
`{1,2,4}`	$(p - r, 0, 0)$	$p - r \geq 0$, $r \geq 0$ *and* $2p + q - 2r \geq 0$
`{2,4,5}`	$(p - r, 2p + q - 2r, 0)$	$p - r \geq 0$, $2p + q - 2r \geq 0$ *and* $r \geq 0$

The chamber is therefore $\{\, (p, q, r) \mid p - r \geq 0 \wedge q \geq 0 \wedge r \geq 0 \,\}$.

Now let us consider general parametric polytopes. First note that we can follow the same procedure as above if we replace **x** by $\mathbf{x}' - \mathbf{c}(\mathbf{p})$ in (5.54), i.e., if our problem has the form

$$\begin{cases} \mathbf{x}' \geq \mathbf{c}(\mathbf{p}) \\ A\,\mathbf{x}' \leq \mathbf{b}(\mathbf{p}) + A\mathbf{c}(\mathbf{p}), \end{cases} \tag{5.56}$$

as saturating a constraint $x_i = 0$ is equivalent to saturating the constraint $x'_i = c_i(\mathbf{p})$ and, similarly, $\langle \mathbf{a}_j, \mathbf{x} \rangle = b_j(\mathbf{p})$ is equivalent to $\langle \mathbf{a}_j, \mathbf{x}' \rangle = b_j(\mathbf{p}) + \langle \mathbf{a}_j, \mathbf{c}(\mathbf{p}) \rangle$.

In the general case, the problem has the form

$$A\mathbf{x} \geq \mathbf{b}(\mathbf{p})$$

and then we apply the technique of subsection 5.16. Let A' be a non-singular square submatrix of A with the same number of columns and compute the (left) HNF $H = A'U$ with U unimodular and H lower-triangular with non-positive elements below the diagonal. Replacing **x** by $U\mathbf{x}'$, we obtain

$$\begin{cases} H\mathbf{x}' \geq \mathbf{b}'(\mathbf{p}) \\ -A''U\,\mathbf{x}' \leq -\mathbf{b}''(\mathbf{p}), \end{cases}$$

with A'' the remaining rows of A and $\mathbf{b}(\mathbf{p})$ split in the same way. If H happens to be the identity matrix, then our problem is of the form (5.56) and we already know how to solve this problem. Note that, again, saturating any of the transformed constraints in $\mathbf{x}'$ is equivalent to saturating the corresponding constraint in **x**. We therefore only need to compute $-A''U$ for the computation of the kernel K. To construct the parametric vertices in the original coordinate system, we can simply use the original constraints. The same reasoning holds if H is any diagonal matrix, since we can virtually replace $H\mathbf{x}$ by $\mathbf{x}'$ without affecting the non-negativity of the variables.

If H is not diagonal, then we can introduce new constraints $x'_j \geq d(\mathbf{p})$, where $d(\mathbf{p})$ is some symbolic constant. These constraints do not remove any solutions since each row in H expresses that the corresponding variable is greater than or equal to a non-negative combination of the previous variables plus some constant. We can then proceed as before. However, to reduce unnecessary computations we may remove from K the rows that correspond to these new rows. Any solution saturating the new constraint, would also saturate the corresponding constraint $\mathbf{h}_j^{\scriptscriptstyle T}$ and all the constraints corresponding to the non-zero entries in $\mathbf{h}_j^{\scriptscriptstyle T}$. If a chamber contains a vertex obtained by saturating such a new constraint, it would appear multiple times in the same chamber, each time combined with different constraints from the above set. Furthermore, there would also be another (as it turns out, identical) chamber where the vertex is only defined by the other constraints.

Example 5.57 *Consider the parametric polytope*

$$P(n) = \{ (i, j) \mid 1 \leq i \wedge 2i \leq 3j \wedge j \leq n \}.$$

The constraints are

$$\begin{bmatrix} 1 & 0 \\ -2 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} \geq \begin{bmatrix} 1 \\ 0 \\ -n \end{bmatrix}.$$

The top 2×2 *submatrix is already in HNF. We have* $3j \geq 2i \geq 2$*, so we can add a constraint of the form* $j \geq c(n)$ *and obtain*

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{bmatrix},$$

while K *with* $\begin{bmatrix} A & I \end{bmatrix} K = 0$ *is given by*

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -2 & 3 \end{bmatrix}.$$

The second row of K *corresponds to the second variable, which in turn corresponds to the newly added constraint. Passing all rows of* K *to* `TOPCOM` *we would get*

```
> points2triangs --regular <<EOF
> [[1 0],[0,1],[0,-1],[-2,3]]
> EOF
T[1]:={{0,1},{0,2},{1,3},{2,3}};
T[2]:={{0,2},{2,3},{0,3}};
T[3]:={};
```

The first vertex in the first chamber saturates the second row (row 1) and therefore saturates both the first (0) and fourth (3) and it appears a second time as `{1,3}`*. Combining these "two" vertices into one as* `{0,3}` *results in the second (identical) chamber. Removing the row corresponding to the new constraint from* K *we remove the duplicates*

```
> points2triangs --regular <<EOF
> [[1 0],[0,-1],[-2,3]]
> EOF
T[1]:={{0,1},{1,2},{0,2}};
T[2]:={};
```

Note that in this example, we also could have interchanged the second and the third constraint and then have replaced j *by* $-j'$*.*

In practice, this method of computing a chamber decomposition does not seem to perform very well, mostly because `TOPCOM` can not exploit all available information about the parametric polytopes and will therefore compute many redundant triangulations/chambers. In particular, any chamber that does not intersect with the parameter domain of the parametric polytope, or only intersects in a face of this parameter domain, is completely redundant. Furthermore, if the parametric polytope is not simple, then many different combinations of the constraints will lead to the same parametric vertex. Many triangulations will therefore correspond to one and the same chamber in the chamber complex of the parametric polytope. For example, for a dilated octahedron, `TOPCOM` will compute 150 triangulations/chambers, 104 of which are empty, while the remaining 46 refer to the same single chamber.

5.18 Computing the Hilbert basis of a cone

To compute the Hilbert basis of a cone, we use the `zsolve` library from `4ti2` (Hemmecke et al.), which implements the technique of Hemmecke (2002). We first remove all equalities from the cone through unimodular transformations and then apply the technique of subsection 5.16 to put the cone in "standard form". Note that for a (non-parametric) cone the constant term $\mathbf{b}$ in (5.51) is $\mathbf{0}$. The constraints $D\mathbf{x} \geq \mathbf{c} = \mathbf{0}$ of (5.52) are therefore equivalent to $\mathbf{x} \geq \mathbf{0}$.

5.19 Integer Feasibility

For testing whether a polytope $P \subset \mathbb{Q}^d$ contains any integer points, we use the technique of Cook et al. (1993), based on generalized basis reduction.

The technique basically looks for a "short vector" $\mathbf{c}$ in the lattice $\mathbb{Z}^d$, where shortness is measured in terms of the width of the polytope P along that direction,

$$\begin{aligned} \operatorname{width}_{\mathbf{c}} P &= \max\{\, \langle \mathbf{c}, \mathbf{x} \rangle \mid \mathbf{x} \in P \,\} - \min\{\, \langle \mathbf{c}, \mathbf{x} \rangle \mid \mathbf{x} \in P \,\} \\ &= \max\{\, \langle \mathbf{c}, \mathbf{x} - \mathbf{y} \rangle \mid \mathbf{x}, \mathbf{y} \in P \,\}. \end{aligned}$$

The *lattice width* is the minimum width over all non-zero integer directions:

$$\operatorname{width} P = \min_{\mathbf{c} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} \operatorname{width}_{\mathbf{c}} P.$$

If the dimension d is fixed then the lattice width of any polytope $P \subset \mathbb{Q}^d$ containing no integer points is bounded by a constant (Lagarias et al. 1990; Barvinok 2002; Banaszczyk et al. 1999). If we slice the polytope using hyperplanes orthogonal to a short direction, i.e., a direction where the width is small, we will therefore only need to inspect "few" of them before either finding one with an integer point, or running out of hyperplanes, meaning that the polytope did not contain any integer points. Each slice is checked for integer points by applying the above method recursively.

A nice feature of this technique is that it will not only tell you if there is any integer point in the given polytope, but it will actually compute one if there is any.

The short vector is obtained as the first vector of a "reduced basis" of the lattice $\mathbb{Z}^d$ with respect to the polytope. In particular, the first vector $\mathbf{b}_1$ of this reduced basis will satisfy

$$\operatorname{width}_{\mathbf{b}_1} P \leq \frac{\operatorname{width} P}{\left(\frac{1}{2} - \varepsilon\right)^{d-1}},$$

with $0 < \varepsilon < 1/2$ a fixed constant. That is, the width in direction $\mathbf{b}_1$ is no more than a constant factor bigger than the lattice width. See (Cook et al. 1993) for details. In our implementation we use $\varepsilon = 1/4$. When used in the above integer feasibility testing algorithm, we will also terminate the reduced basis computation as soon as the width along the first basis vector is smaller than 2. This means that there will be at most 2 slices orthogonal to the chosen direction.

The computation of the above reduced basis requires the solution of many linear programs, for which we use any of the following external solvers:

- `GLPK` (Makhorin 2006)

  This solver is based on double precision floating point arithmetic and may therefore not be suitable if the coefficients of the constraints describing the polytope are large.

- `cdd` (Fukuda 1993)

  This solver is based on exact integer arithmetic. Note that you need version `cddlib 0.94e` or newer. Earlier versions (`0.93`–`0.94d`) have a bug that may sometimes result in a polytope being reported as (rationally) empty even though it is not.

- `piplib` (Feautrier 2006)

  This solver is also based on exact integer arithmetic and uses the dual simplex method to solve a linear program. Two versions are available, `pip` will present the original program to `piplib`, while `pip-dual` will present the dual program to `piplib`, effectively having it apply the primal simplex method to the original problem. The latter may seem more appropriate since the computation of the reduced basis only requires the dual solution of any linear program. However, in practice, it appears that `pip` is often faster than `pip-dual`.

The LP solver to use can be selected with the `--gbr` option.

5.20 Computing the integer hull of a polyhedron

For computing the integer hull of a polyhedron, we first describe how to compute the convex hull of a set given as an oracle for optimizing a linear objective function over the set and then we explain how to optimize a linear objective function over the integer points of a polyhedron. Applying the first with the second as optimization oracle yields a method for computing the requested integer hull.

5.20.1 Computing the convex hull based on an optimization oracle

The algorithm described below is presented by Cook et al. (1992, Remark 2.5) as an extension of the algorithm by Edmonds et al. (1982, Section 3) for computing the *dimension* of a polytope for which only an optimization oracle is available. The algorithm is described in a bit more detail by Eisenbrand (2000) and reportedly stems from Hartmann (1989). Essentially the same algorithm has also been implemented by Huggins (2006), citing beneath/beyond (Preparata and Shamos 1985) as his inspiration.

The algorithm start out from an initial set of points from the set S. After computing the convex hull of this set of points, we take one of its bounding constraints and use the optimization oracle to compute an optimal point in S (but on the other side of the bounding hyperplane) along the outer normal of this bounding constraint. If a new point is found, it is added to the set of points and a new convex hull is computed, or the old one is adapted in a beneath/beyond fashion. Otherwise, the chosen bounding constraint is also a bounding constraint of S and need not be considered anymore. The

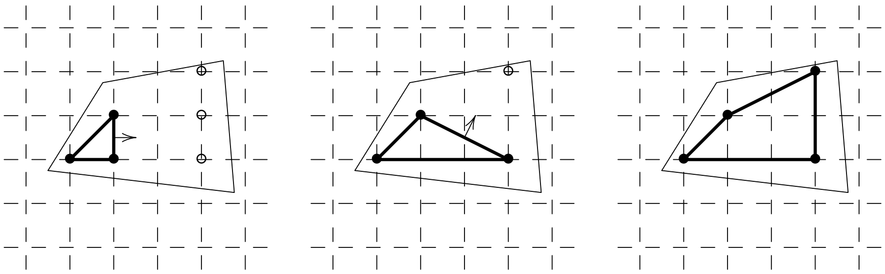


Figure 5.58: The integer hull of a polytope

process continues until all bounding constraints in the description of the current convex hull have been considered.

In principle, the initial set of points in the above algorithm may be empty, with a "convex hull" described by a set of conflicting constraints and each equality in the description of any intermediate lower-dimensional convex hull being considered as a pair of bounding constraints with opposite outer normals. However, in our implementation, we have chosen to first compute a maximal set of affinely independent points by first taking any point from S and then adding points from S not on one of the equalities satisfied by all points found so far. This allows us to not have to worry about equalities in the main algorithm. In the case of the computation of the integer hull, finding these affinely independent points can be accomplished using the technique of subsection 5.19.

Example 5.59 *Assume we want to compute the integer hull of the polytope in the left part of Figure 5.58. We first compute a set of three affinely independent points, shown in the same part of the figure. Of the three facets of the corresponding convex hull, optimization along the outer normal (depicted by an arrow in the figure) of only one facet will yield any additional points. The other two are therefore facets of the integer hull. Optimization along the above outer normal may yield any of the points marked by a* $\circ$*. Assuming it is the bottom one, we end up with the updated convex hull in the middle of the figure. This convex hull has only one new facet. Adding the point found by optimizing over this facet's outer normal, we obtain the convex hull on the right of the figure. There are two new facets, but neither of them yields any further points. We have therefore found the integer hull of the polytope.*

5.20.2 Optimization over the integer points of a polyhedron

We assume that we want to find the *minimum* of some linear objective function. When used in the computation of the integer hull of some polytope, the objective function will therefore correspond to the inner normal of some facet.

During our search for an optimal integer point with respect to some objective function, we will keep track of the best point so far as well as a lower bound l and an upper bound u such that the value at the optimal point (if it is better than the current best) lies between those two bounds. Initially, there is no best point yet and values for l and u

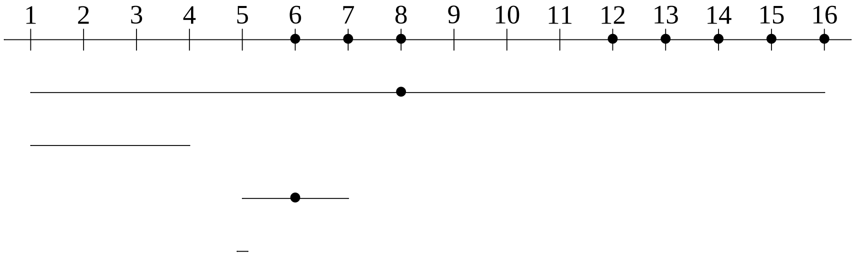


Figure 5.60: The integer points of a polytope projected on an objective function

may be obtained from optimization over the linear relaxation. When used in the computation of the integer hull of some polytope, the upper bound u is one less than the value attained on the given facet of the current approximation.

As long as $l \leq u$, we perform the following steps

- use the integer feasibility technique of subsection 5.19 to test whether there is any integer point with value in $[l, u']$, where u' is
  - u if the previous test for an integer point did not produce a point
  - $l + \left\lfloor \frac{u-l-1}{2} \right\rfloor$ if the previous test for an integer point *did* produce a point
- if a point is found, then remember it as the current best and replace u by the value at this point minus one,
- otherwise, replace l by $u' + 1$.

When used in the computation of the integer hull of some polytope, it is useful to not only keep track of the best point so far, but of all points found. These points will all lie outside of the current approximation of the integer hull and adding them all instead of just one, will typically get us to the complete integer hull quicker.

Example 5.61 *Assume that the values of some objective function attained by the integer points of some polytope are as shown in Figure 5.60 and assume we know that the optimal value lies between 1 and 16. In the first step we would look for a point attaining a value in the interval* $[1, 16]$. *Suppose this yields a point attaining the value* 8 *(second line of the figure). We record this point as the current best and update the search interval to* $[1, 7]$. *In the second step, we look for a point attaining a value in the interval* $[1, 4]$, *but find nothing and set the search interval to* $[5, 7]$. *In the third step, we consider the interval* $[5, 7]$ *and find a point attaining the value 6. We update the current best value and set the search interval to* $[5, 5]$. *In the fourth step, we consider the interval* $[5, 5]$, *find no points and update the interval to* "$[6, 5]$". *Since the lower bound is now larger than the upper bound, the algorithm terminates, returning the best or all point(s) found.*

5.21 Computing the integer hull of a truncated cone

In subsection 5.22 we will need to compute the integer hull of a cone with the origin removed ($C \setminus \{\mathbf{0}\}$).

5.21.1 Using the Hilbert basis of the cone

As proposed by Köppe (2007), one way of computing this integer hull is to first compute the Hilbert basis of C (see subsection 5.18) and to then remove from that Hilbert basis the points that are not vertices of the integer hull of $C \setminus \{\mathbf{0}\}$. The Hilbert basis of C is the minimal set of points $\mathbf{b}_i \in C \cap \mathbb{Z}^d$ such that every integer point $\mathbf{x} \in C \cap \mathbb{Z}^d$ can be written as a non-negative *integer* combination of the $\mathbf{b}_i$. The vertices $\mathbf{v}_j$ of the integer hull of $C \setminus \{\mathbf{0}\}$ are such that every integer point $\mathbf{x} \in (C \cap \mathbb{Z}^d) \setminus \{\mathbf{0}\}$ can be written as s non-negative *rational* combination of $\mathbf{v}_j$. Clearly, any $\mathbf{v}_j$ is also a $\mathbf{b}_i$ since $\mathbf{v}_j$ can not be written as the sum of a (rational) convex combination of other integer points in $(C \cap \mathbb{Z}^d) \setminus \{\mathbf{0}\}$ and a non-negative combination of the extremal rays $\mathbf{r}_k$ of C. A fortiori, it can therefore not be written as an integer combination of other integer points in C. To obtain the $\mathbf{v}_j$ from the $\mathbf{b}_i$ we therefore simply need to remove first $(0,0)$ and then those $\mathbf{b}_i$ that are not an extremal ray and that *can* be written as a combination

$$\mathbf{b}_i = \sum_{j \ne i} \alpha_j \mathbf{b}_j + \sum_k \beta_k \mathbf{r}_k \qquad \text{with } \alpha_j, \beta_k \ge 0 \text{ and } \sum_{j \ne i} \alpha_j = 1.$$

Since the $\mathbf{r}_k$ are also among the $\mathbf{b}_j$, this can be simplified to checking whether there exists a rational solution for α_j to

$$\mathbf{b}_i = \sum_{j \ne i} \alpha_j \mathbf{b}_j \qquad \text{with } \alpha_j \ge 0 \text{ and } \sum_{j \ne i} \alpha_j \ge 1.$$

Example 5.63 *Consider the cone*

$$C = \operatorname{pos}\{(2,-3),(3,4)\},$$

shown in Figure 5.62. The Hilbert basis of this cone is

$$\{(0,0),(2,-3),(3,4),(1,1),(1,-1),(1,0)\}.$$

We have $(1,0) = \frac{1}{2}(1,1) + \frac{1}{2}(1,-1)$*, while* $(1,1)$ *and* $(1,-1)$ *can not be written as overconvex combinations of the other* $\mathbf{b}_i \ne \mathbf{0}$*. The vertices of the integer hull of* $C \setminus \{\mathbf{0}\}$ *are therefore*

$$\{(2,-3),(3,4),(1,1),(1,-1)\}.$$

5.21.2 Using generalized basis reduction

Another way of computing the integer hull of a truncated cone is to apply the method of subsection 5.20. In this case, the initial set of points will consist of (the smallest integer representatives of) the extremal rays of the cone, together with the extremal

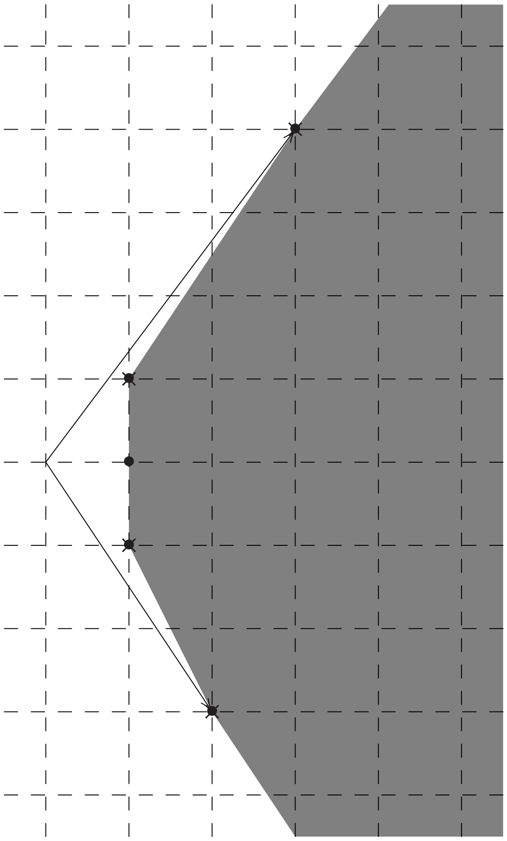


Figure 5.62: The Hilbert basis and the integer hull of a truncated cone

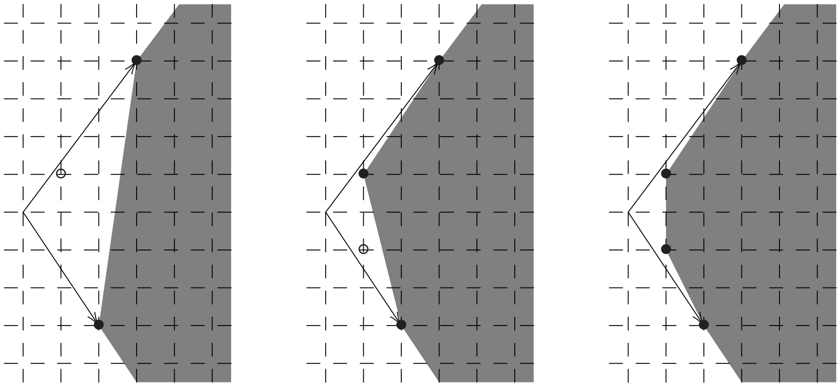


Figure 5.64: The integer hull of a truncated cone

rays themselves. That is, if $C = \operatorname{pos}\{\mathbf{r}_j\}$ with $\mathbf{r}_j \in \mathbb{Z}^d$, then our initial approximation of the integer hull of $C \setminus \{\mathbf{0}\}$ is

$$\operatorname{conv}\{\mathbf{r}_j\} + \operatorname{pos}\{\mathbf{r}_j\}.$$

Furthermore, we need never consider any of the bounding constraints that are also bounding constraints of the original cone. When optimizing along the normal of any of the other facets, we can take the lower bound to be 1. This will ensure that the origin is excluded, without excluding any other integer points.

Example 5.65 *Consider once more the cone*

$$C = \operatorname{pos}\{(2,-3),(3,4)\}$$

from Example 5.63. The initial approximation is

$$C = \operatorname{conv}\{(2,-3),(3,4)\} + \operatorname{pos}\{(2,-3),(3,4)\},$$

which is shown on the left of Figure 5.64. The only bounding constraint that does not correspond to a bounding constraint of C is $7x - y \geq 17$*. In the first step, we will therefore look for a point minimizing* $7x - y$ *with values in the interval* $[1,16]$*. All values of this objective function in the given interval attained by points in C are shown in Figure 5.60. From Example 5.61, we know that the optimal value is* 6 *and this corresponds to the point* $(1,1)$*. Adding this point to our hull, we obtain the approximation in the middle of Figure 5.64. This approximation has two new facets. The bounding constraint* $3x - 2y \geq 1$ *will not produce any new points since we would be looking for one in the interval "*$[1,0]$*". The other new bounding constraint is* $4x + y \geq 5$*. Minimizing* $4x + y$ *with values in the interval* $[1,4]$*, we find the minimal value* 3 *corresponding to the point* $(1,-1)$*. Adding this point, we obtain the complete integer hull shown on the right of Figure 5.64. Note that if in the first step we would have added not only the point corresponding to the optimal value, but instead all points found in Example 5.61, then we would have obtained the complete integer hull directly.*

5.22 Computing the lattice width of a parametric polytope

To compute the lattice width of a parametric polytope, we essentially use the technique of Eisenbrand and Shmonin (2007), which improves upon the technique of Kannan (1992). Given a parametric polytope

$$P(\mathbf{p}) = \{\, \mathbf{x} \mid A\mathbf{x} + \mathbf{b}(\mathbf{p}) \geq \mathbf{0} \,\},$$

the width along a direction $\mathbf{c}$ is defined in the same way as for non-parametric polytopes (see subsection 5.19),

$$\operatorname{width}_{\mathbf{c}} P(\mathbf{p}) = \max\{\, \langle \mathbf{c}, \mathbf{x} \rangle \mid \mathbf{x} \in P(\mathbf{p}) \,\} - \min\{\, \langle \mathbf{c}, \mathbf{x} \rangle \mid \mathbf{x} \in P(\mathbf{p}) \,\}. \tag{5.66}$$

The *lattice width* is the minimum width over all non-zero integer directions:

$$\operatorname{width} P(\mathbf{p}) = \min_{\mathbf{c} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} \operatorname{width}_{\mathbf{c}} P(\mathbf{p}).$$

We assume that the parameter domain Q of $P(\mathbf{p})$, i.e., the set of parameter values for which $P(\mathbf{p}) \neq \emptyset$, is full-dimensional and that for each $\mathbf{p}$ from the interior of Q, $P(\mathbf{p})$ is also full-dimensional.

Clearly, for any given direction $\mathbf{c}$, the minimum and maximum in (5.66) are attained at (different) vertices of $P(\mathbf{p})$. The idea of the algorithm is then to consider all pairs of parametric vertices of $P(\mathbf{p})$, to compute all candidate integer directions for a given pair of vertices and then to compute the minimum width over all candidate integer directions found.

For any given parametric vertex $\mathbf{v}(\mathbf{p})$, the (rational) directions for which this vertex is minimal can be found as follows. Let $\mathbf{v}(\mathbf{p}) + C$ be the vertex cone of $\mathbf{v}(\mathbf{p})$. If $\mathbf{v}(\mathbf{p})$ is minimal for $\mathbf{c}$, then all other points in the vertex cone must yield a bigger or equal value, i.e., $\langle \mathbf{y}, \mathbf{c} \rangle \geq 0$ for all $\mathbf{y} \in C$. The set of directions is therefore the polar cone C^* of C. Note that, in principle, we should only do this for pairs of vertices that have a common activity domain, where the activity domains have been partially opened using the technique of Theorem 5.12 to avoid multiple vertices that coincide on a lower-dimensional chamber to all be considered on this intersection. However, this optimization has currently not been implemented.

Given a pair of vertices $\mathbf{v}_1(\mathbf{p})$ and $\mathbf{v}_2(\mathbf{p})$, we may assume that $\mathbf{v}_1(\mathbf{p})$ attains the minimum and $\mathbf{v}_2(\mathbf{p})$ attains the maximum. If $\mathbf{v}_1(\mathbf{p})+C_1$ and $\mathbf{v}_2(\mathbf{p})+C_2$ are the corresponding vertex cones, then the set of (rational) directions for this pair of vertices is

$$C_{1,2} = (C_1^* \cap -C_2^*) \setminus \{\mathbf{0}\}.$$

The set of candidate integer directions are therefore the vertices of the integer hull of $C_{1,2}$, which can be computed as explained in subsection 5.21. To see this, note that by construction $\langle \mathbf{c}, \mathbf{v}_1(\mathbf{p}) \rangle \leq \langle \mathbf{c}, \mathbf{v}_2(\mathbf{p}) \rangle$ and so

$$w_{\mathbf{c}}(\mathbf{p}) = \operatorname{width}_{\mathbf{c}} P(\mathbf{p}) = \langle \mathbf{c}, \mathbf{v}_2(\mathbf{p}) - \mathbf{v}_1(\mathbf{p}) \rangle \geq 0.$$

Any integer direction in $C_{1,2}$ will therefore yield a width that is at least as large as that of one of the vertices of the integer hull. Note that when using generalized basis

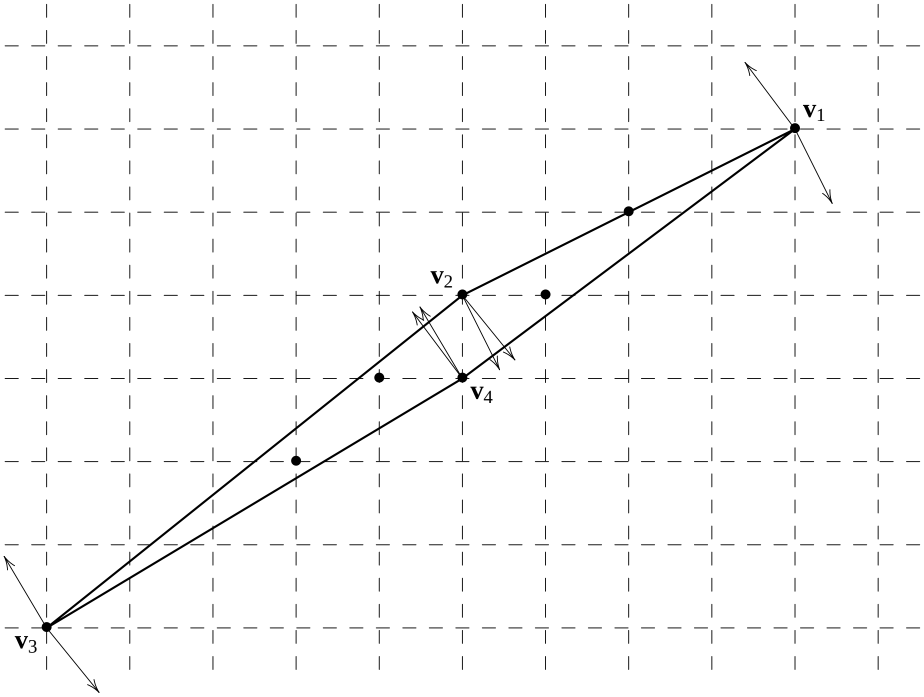


Figure 5.67: A polytope and its candidate width directions

reduction to compute the integer hull of these cones as in subsubsection 5.21.2, it can be helpful to use as vertices for the initial approximation not only the extremal rays of the cone, but also those vertices of previously computed integer hulls that are elements of the current cone.

After computing a list of all possible candidate width directions $\mathbf{c}_i$ and the corresponding widths $w_{\mathbf{c}_i}(\mathbf{p})$, we keep only a single direction of all those that yield the same width (as an affine function of the parameters). Then we construct the chambers where each of the widths is minimal, i.e.,

$$C_i = \{\, \mathbf{p} \in Q \mid \forall j : w_{\mathbf{c}_i}(\mathbf{p}) \le w_{\mathbf{c}_j}(\mathbf{p}) \,\}.$$

Note that many of the C_i may be empty or of lower dimension than Q and that the other C_i will intersect in common facets. To obtain a partition of partially-open full-dimensional chambers, we proceed as in subsection 5.4.

Example 5.68 *Consider the (non-parametric) polytope*

$$P = \left\{ \mathbf{x} \mid \begin{array}{r} -3x_1 + 5x_2 \ge 0 \\ 4x_1 - 5x_2 \ge 0 \\ x_1 - 2x_2 + 3 \ge 0 \\ -3x_1 + 4x_2 + 3 \ge 0 \end{array} \right\}$$

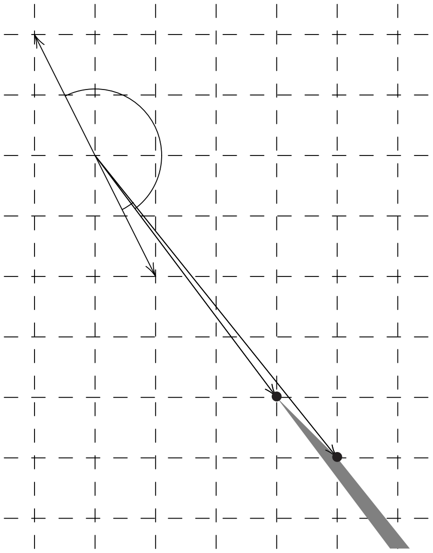


Figure 5.69: The cone of directions $C_{2,1}$

shown in Figure 5.67. The polytope has four vertices

$$\begin{aligned} \mathbf{v}_1 &= (9,6) \\ \mathbf{v}_2 &= (5,4) \\ \mathbf{v}_3 &= (0,0) \\ \mathbf{v}_4 &= (5,3). \end{aligned}$$

The corresponding cones of directions (for the given vertex to attain the minimum), also shown in Figure 5.67 are

$$\begin{aligned} C_1^* &= \operatorname{pos}\{(-3,4),(1,-2)\} \\ C_2^* &= \operatorname{pos}\{(4,-5),(1,-2)\} \\ C_3^* &= \operatorname{pos}\{(4,-5),(-3,5)\} \\ C_4^* &= \operatorname{pos}\{(-3,5),(-3,4)\}. \end{aligned}$$

Let us now consider the directions in which $\mathbf{v}_2$ *is minimal while* $\mathbf{v}_1$ *is maximal. We find*

$$C_{2,1} = \operatorname{pos}\{(4,-5),(3,-4)\} \setminus \{\mathbf{0}\},$$

as shown in Figure 5.69. The vertices of the integer hull of $C_{2,1}$ *are* $(4,-5)$ *and* $(3,-4)$. *The corresponding widths are*

$$\begin{aligned} \mathbf{c}_1 &= (4,-5) \quad w_{\mathbf{c}_1} = 6 \\ \mathbf{c}_2 &= (3,-4) \quad w_{\mathbf{c}_2} = 4. \end{aligned}$$

We similarly find

$$C_{3,1} = \operatorname{pos}\{(4,-5),(-1,2)\} \setminus \{\mathbf{0}\},$$

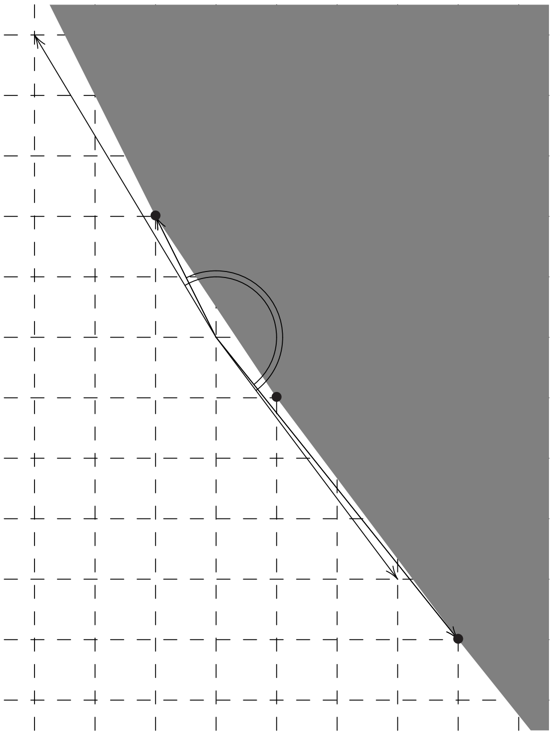


Figure 5.70: The cone of directions $C_{3,1}$

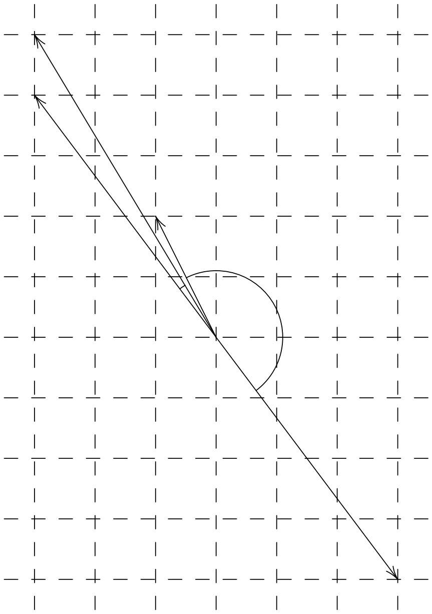


Figure 5.71: The cone of directions $C_{4,1}$

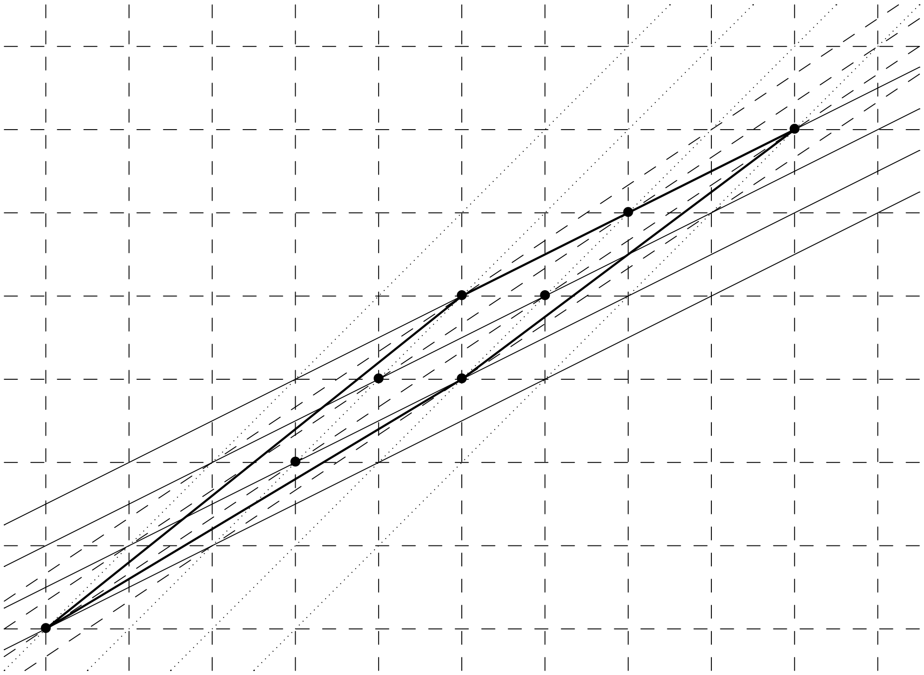


Figure 5.72: A polytope and its lattice width directions

with integer hull $\operatorname{pos}\{(4,-5),(-1,2),(1,-1)\}$, *shown in Figure 5.70, yielding*

$$\begin{aligned}\mathbf{c}_3 &= (4,-5) \quad w_{\mathbf{c}_3} = 6\\ \mathbf{c}_4 &= (-1,2) \quad w_{\mathbf{c}_4} = 3\\ \mathbf{c}_5 &= (1,-1) \quad w_{\mathbf{c}_5} = 3.\end{aligned}$$

On the other hand,

$$C_{4,1} = \emptyset,$$

as shown in Figure 5.71 and so this combination does not yield any width direction candidates. The other pairs of vertices further yield

$$\begin{aligned}\mathbf{c}_6 &= (-1,2) \quad w_{\mathbf{c}_6} = 3\\ \mathbf{c}_7 &= (-3,5) \quad w_{\mathbf{c}_7} = 5\\ \mathbf{c}_8 &= (-3,4) \quad w_{\mathbf{c}_8} = 4\\ \mathbf{c}_9 &= (-3,5) \quad w_{\mathbf{c}_9} = 5\\ \mathbf{c}_{10} &= (-2,3) \quad w_{\mathbf{c}_{10}} = 3.\end{aligned}$$

Since the polytope under consideration is not parametric, there is only one (non-empty, 0*-dimensional) chamber and it corresponds to one of the directions, say* $\mathbf{c}_4 = (-1,2)$, *with width* 3 *(the other directions with the same width having been removed).*

Each of the three directions that yield the minimal width of 3 is shown in Figure 5.72.

Example 5.73 *Consider the polytope*

$$P(p) = \left\{ \mathbf{x} \mid \begin{array}{r} -2x_1 + p + 5 \geq 0 \\ 2x_1 + p + 5 \geq 0 \\ -2x_2 - p + 5 \geq 0 \\ 2x_2 - p + 5 \geq 0 \end{array} \right\}$$

from Woods (2004, Example 2.1.7). The parametric vertices are

$$\begin{aligned} \mathbf{v}_1(p) &= \left(\frac{p+5}{2}, \frac{-p+5}{2} \right) \\ \mathbf{v}_2(p) &= \left(\frac{p+5}{2}, \frac{p-5}{2} \right) \\ \mathbf{v}_3(p) &= \left(\frac{-p-5}{2}, \frac{-p+5}{2} \right) \\ \mathbf{v}_4(p) &= \left(\frac{-p-5}{2}, \frac{p-5}{2} \right). \end{aligned}$$

We find two essentially different candidate width directions

$$\begin{aligned} \mathbf{c}_1 &= (0,1) \quad w_{\mathbf{c}_1}(p) = 5 - p \\ \mathbf{c}_2 &= (1,0) \quad w_{\mathbf{c}_2}(p) = 5 + p. \end{aligned}$$

The first direction can be found by combining, say, $\mathbf{v}_1(p)$ *and* $\mathbf{v}_2(p)$*, while the second direction can be found by combining, say,* $\mathbf{v}_1(p)$ *and* $\mathbf{v}_3(p)$*. The parameter domain for the parametric polytope* $P(p)$ *is*

$$Q = \{ p \mid -5 \leq p \leq 5 \}.$$

The two (closed) chambers are therefore

$$\begin{aligned} C_1 &= \{ p \in Q \mid 5 - p \leq 5 + p \} \\ C_2 &= \{ p \in Q \mid 5 + p \leq 5 - p \}. \end{aligned}$$

To obtain a partition, subsection 5.1 gives the internal point $(0,0)$*, which happens to meet the facets* $p \geq 0$ *and* $-p \geq 0$*. We therefore keep the facet with positive (inner) normal closed and open up the other facet. The result is*

$$\begin{aligned} \hat{C}_1 &= \{ p \mid 0 \leq p \leq 5 \} \\ \hat{C}_2 &= \{ p \mid -5 \leq p < 0 \}. \end{aligned}$$

Since we are usually only interested in integer parameter values, the latter chamber would become $\hat{C}_2 = \{ p \mid -5 \leq p \leq -1 \}$.

Our description differs slightly from that of of Eisenbrand and Shmonin (2007). First, they consider all pairs of basic solutions instead of all pairs of vertices, which means that they may consider basic solutions that are never feasible and that, in case of a non-simple polytope, they may consider the same parametric vertex more than once. The set of integer directions for a pair of vertices is the intersection of the sets of integer directions they obtain for each of the corresponding basic solutions. Second, they use a different method of creating a partition of partially-open chambers, which may lead to some lower-dimensional chambers surviving and hence to a larger total number of chambers.

5.23 Testing whether a set has an infinite number of points

In some situations we are given the generating function of some integer set and we would like to know if the set is infinite or not. Typically, we want to know if the set is empty or not, but we cannot simply count the number of elements in the standard way since we may not have any guarantee that the set has only a finite number of elements. We will consider the slightly more general case where we are given a rational generating function $f(\mathbf{x})$ of the form (5.26) such that

$$f(\mathbf{x}) = \sum_{\mathbf{s}\in Q\cap\mathbb{Z}^d} c(\mathbf{s})\,\mathbf{x}^{\mathbf{s}} \tag{5.74}$$

converges on some nonempty open subset of $\mathbb{C}^d$, Q is a pointed polyhedron and $c(\mathbf{s}) \geq 0$, and we want to compute

$$S = \sum_{\mathbf{s}\in Q\cap\mathbb{Z}^d} c(\mathbf{s}), \tag{5.75}$$

where the sum may diverge, i.e., "$S = \infty$". The following proposition shows that we can determine S in polynomial time. For a sketch of an alternative technique, see Woods (2005, Proof of Lemma 16).

Proposition 5.76 *Fix d and k. Given a rational generating function of the form* (5.26) *with $k_i \leq k$ and a pointed polyhedron $Q \subset \mathbb{Q}^d$, then there is a polynomial time algorithm that determines for the corresponding function* $c(\mathbf{s})$ (5.74) *whether the sum* (5.75) *diverges and computes the value of S* (5.75) *if it does not.*

Proof Since Q is pointed, the series (5.74) converges on a neighborhood of $e^{\boldsymbol{\ell}} = (e^{\ell_1}, \ldots, e^{\ell_d})$ for any $\boldsymbol{\ell}$ such that $\langle \mathbf{r}_k, \boldsymbol{\ell}\rangle < 0$ for any (extremal) ray $\mathbf{r}_k$ of Q and such that $\langle \mathbf{b}_{ij}, \boldsymbol{\ell}\rangle \neq 0$ for any $\mathbf{b}_{ij}$ in (5.26). Let $\boldsymbol{\alpha} = -\boldsymbol{\ell}$ and perform the substitution $\mathbf{x} = t^{\boldsymbol{\alpha}}$. The function $g(t) = f(t^{\boldsymbol{\alpha}})$ is then also a (short) rational generating function and

$$g(t) = \sum_{k\in\langle\boldsymbol{\alpha},Q\rangle\cap\mathbb{Z}} \left(\sum_{\substack{\mathbf{s}\in Q\cap\mathbb{Z}^d \\ \langle\boldsymbol{\alpha},\mathbf{s}\rangle=k}} c(\mathbf{s}) \right) t^k =: \sum_{k\in\langle\boldsymbol{\alpha},Q\rangle\cap\mathbb{Z}} d(k)\, t^k,$$

with $\langle\boldsymbol{\alpha}, Q\rangle = \{\langle\boldsymbol{\alpha}, \mathbf{x}\rangle \mid \mathbf{x} \in Q\}$, converges in a neighborhood of e^{-1}, while

$$S = \sum_{k\in\langle\boldsymbol{\alpha},Q\rangle\cap\mathbb{Z}} d(k).$$

Since $c(\mathbf{s}) \geq 0$, we have $d(k) \geq 0$ and the above sum diverges iff any of the coefficients of the negative powers of t in the Laurent expansion of $g(t)$ is non-zero. If the sum converges, then the sum is simply the coefficient of the constant term in this expansion.

It only remains to show now that we can compute a suitable $\boldsymbol{\alpha}$ in polynomial time, i.e., an $\boldsymbol{\alpha}$ such that $\langle \mathbf{r}_k, \boldsymbol{\alpha} \rangle > 0$ for any (extremal) ray $\mathbf{r}_k$ of Q and $\langle \mathbf{b}_{ij}, \boldsymbol{\alpha} \rangle \neq 0$ for any $\mathbf{b}_{ij}$ in (5.26). By adding the $\mathbf{r}_k$ to the list of $\mathbf{b}_{ij}$ if needed, we can relax the first set of constraints to $\langle \mathbf{r}_k, \boldsymbol{\alpha} \rangle \geq 0$. Let Q be described by the constraints $A\mathbf{x} + \mathbf{c} \geq \mathbf{0}$ and let B be $d \times d$ non-singular submatrix of A, obtained by removing some of the rows of A. Such a B exists since Q does not contain any straight line. Clearly, $B\mathbf{r} \geq \mathbf{0}$ for any ray $\mathbf{r}$ of Q. Let $\mathbf{b}'_{ij} = B\mathbf{b}_{ij}$, then since $\mathbf{b}_{ij} \neq \mathbf{0}$ and B is non-singular, we have $\mathbf{b}'_{ij} \neq \mathbf{0}$. We may therefore find in polynomial time a point $\boldsymbol{\alpha}' \geq \mathbf{0}$ on the "moment curve" such that $\langle \mathbf{b}'_{ij}, \boldsymbol{\alpha}' \rangle \neq 0$ (Barvinok and Pommersheim 1999, Algorithm 5.2). Let $\boldsymbol{\alpha} = B^{\scriptscriptstyle T}\boldsymbol{\alpha}'$. Then $\langle \mathbf{b}_{ij}, \boldsymbol{\alpha} \rangle = \langle \mathbf{b}_{ij}, B^{\scriptscriptstyle T}\boldsymbol{\alpha}' \rangle = \langle B\mathbf{b}_{ij}, \boldsymbol{\alpha}' \rangle = \langle \mathbf{b}'_{ij}, \boldsymbol{\alpha}' \rangle \neq 0$ and $\langle \mathbf{r}_k, \boldsymbol{\alpha} \rangle = \langle \mathbf{r}_k, B^{\scriptscriptstyle T}\boldsymbol{\alpha}' \rangle = \langle B\mathbf{r}_k, \boldsymbol{\alpha}' \rangle \geq 0$, as required. Note that in practice, we would, as usual, first try a fixed number of random vectors $\boldsymbol{\alpha}' \geq \mathbf{0}$ before resorting to looking for a point on the moment curve.

□

5.24 Enumerating integer projections of parametric polytopes

In this section we are interested in computing

$$c(\mathbf{s}) = \#\left\{ \mathbf{t} \in \mathbb{Z}^d \mid \exists \mathbf{u} \in \mathbb{Z}^m : (\mathbf{s}, \mathbf{t}, \mathbf{u}) \in P \right\}, \tag{5.77}$$

with $P \subset \mathbb{Q}^n \times \mathbb{Q}^d \times \mathbb{Q}^m$ a rational pointed polyhedron such that

$$P_{\mathbf{s}} = \left\{ (\mathbf{t}, \mathbf{u}) \in \mathbb{Q}^d \times \mathbb{Q}^m \mid (\mathbf{s}, \mathbf{t}, \mathbf{u}) \in P \right\}$$

is a polytope for any $\mathbf{s}$. This is equivalent to computing the number of points in the integer projection of a parametric polytope

$$c(\mathbf{s}) = \#(\pi(P_{\mathbf{s}} \cap \mathbb{Z}^{d+m})),$$

with $\pi : \mathbb{Q}^d \times \mathbb{Q}^m \to \mathbb{Q}^d$ defined by $\pi(\mathbf{t}, \mathbf{u}) = \mathbf{t}$. Exponential methods for computing $c(\mathbf{s})$ are described by Verdoolaege et al. (2005a) and Seghir and Loechner (2006). Here, we provide some implementation details for the polynomial method of Barvinok and Woods (2003, Theorem 1.7), for computing the generating function, $\sum_{\mathbf{s}} c(\mathbf{s})\, \mathbf{x}^{\mathbf{s}}$, which can then be converted into an explicit function $c(\mathbf{s})$ (Verdoolaege and Woods 2008, Corollary 1.11). Note that in contrast to Barvinok and Woods (2003, Theorem 1.7), we may allow P to be an unbounded (but still pointed) polyhedron here (as long as $P_{\mathbf{s}}$ is bounded), since we replace their application of Kannan (1992, Lemma 3.1) by Eisenbrand and Shmonin (2007, Theorem 5).

If there is only one existentially quantified variable ($m = 1$), then computing (5.77) is easy. You simply shift P by 1 in the u direction and subtract this shifted copy from the original,

$$D = P \setminus (\mathbf{e}_{n+d+1} + P).$$

(See, e.g., Barvinok and Woods (2003, Figure 1, page 973) or Verdoolaege (2005, Figure 4.33, page 186).) In the difference D there will be *exactly* one value of u for each value of the remaining variables for which there was *at least* one value of u in P,

$$\forall(\mathbf{s},\mathbf{t}) : \quad (\exists u : (\mathbf{s},\mathbf{t},u) \in P) \iff (\exists! u : (\mathbf{s},\mathbf{t},u) \in D)\,.$$

The function $c(\mathbf{s})$ can then be computed by counting the number of elements in $D(\mathbf{s})$. These operations can be performed either in the space of (unions of) parametric polytopes or on generating functions. In the first case, $D(\mathbf{s})$ can be written as a disjoint union of parametric polytopes that can be enumerated separately. In the second case, we first compute the generating function $f(\mathbf{x},\mathbf{y})$ of the set

$$S = \{(\mathbf{s},\mathbf{t}) \mid \exists u \in \mathbb{Z} : (\mathbf{s},\mathbf{t},u) \in P\}$$

and then obtain the generating function $C(\mathbf{x})$ of $c(\mathbf{s})$ as $C(\mathbf{x}) = f(\mathbf{x},\mathbf{1})$. In the remainder of this section, we will concentrate on the computation of the generating function of S. To compute this generating function in the current case where there is only one existentially quantified variable, we first compute the generating function $g(\mathbf{x},\mathbf{y},z)$ of $P(\mathbf{s},\mathbf{t},u)$, perform operations on the generating function equivalent to the set operations (see, e.g., Verdoolaege (2005, Section 4.5.3)), resulting in a generating function $g'(\mathbf{x},\mathbf{y},z)$, and then sum over all values (at most one for each value of $\mathbf{s}$ and $\mathbf{t}$) of u, i.e., $f(\mathbf{x},\mathbf{y}) = g'(\mathbf{c},\mathbf{y},1)$.

If there is more than one existentially quantified variable ($m > 1$), then we can in principle apply the above shifting and subtracting technique recursively to obtain a generating function $f(\mathbf{x},\mathbf{y})$ for the set

$$T = \{(\mathbf{s},\mathbf{t}) \mid \exists \mathbf{u} \in \mathbb{Z}^m : (\mathbf{s},\mathbf{t},\mathbf{u}) \in P\} \tag{5.79}$$

and then compute $C(\mathbf{x}) = f(\mathbf{x},\mathbf{1})$. There are however some complications. Most notably, after applying the technique in one direction and projecting out the corresponding variable, the resulting set, i.e.,

$$S = \{(\mathbf{s},\mathbf{t},u_1,\ldots,u_{m-1}) \mid \exists u_m \in \mathbb{Z} : (\mathbf{s},\mathbf{t},\mathbf{u}) \in P\},$$

in general does not correspond to the integer points in some polytope. For example, assume that the polytope in Figure 5.78 contains the values of $\mathbf{u}$ associated to a particular value of $(\mathbf{s},\mathbf{t})$. Since there are integer points in this polytope, we should count this value of $\mathbf{t}$, but only once. If we apply the above technique in the vertical direction (u_2), then we can compute (a generating function for) the set S shown on the bottom of the figure. Note, however, that there are "gaps" in this set, i.e., if we compute $S \setminus (\mathbf{e}_{n+d+1} + S)$ then we will not end up with a single point (for this value of $(\mathbf{s},\mathbf{t})$). Since the biggest gap is three wide, we need to compute

$$S \setminus (\mathbf{e}_{n+d+1} + S) \setminus (2\mathbf{e}_{n+d+1} + S) \setminus (3\mathbf{e}_{n+d+1} + S)$$

to obtain a single point. If we do the subtraction in the horizontal direction first, then we end up with a set (shown on the left) with gaps at most two wide, so afterwards we only need to subtract twice in the vertical direction.

In general, there is no bound on the widths of the gaps we may encounter in any given direction. However, there are directions in which the gaps are known to

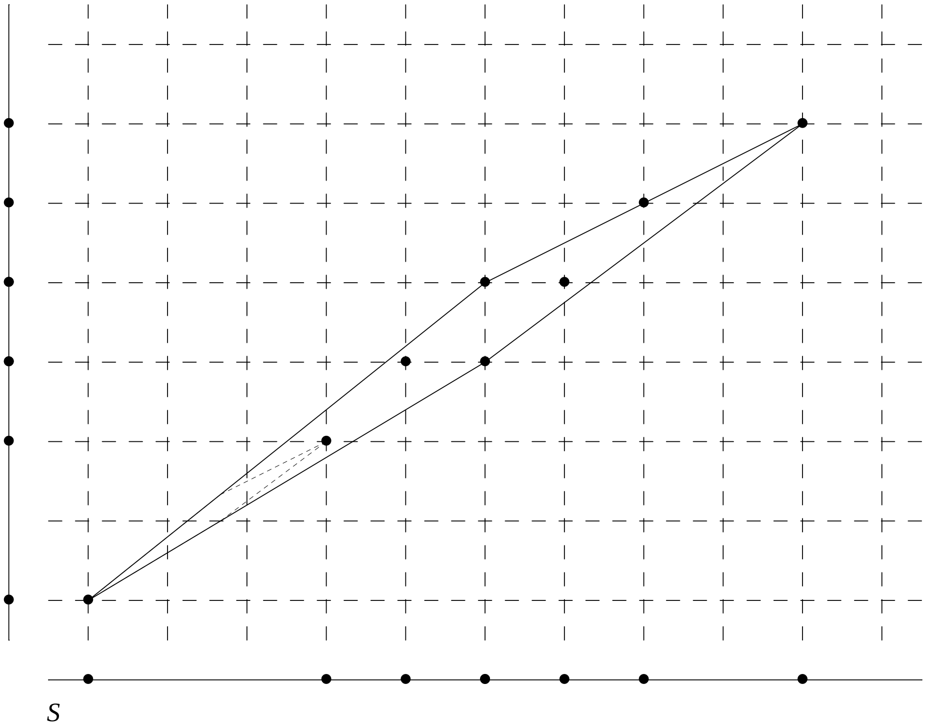


Figure 5.78: A polytope and its integer projections

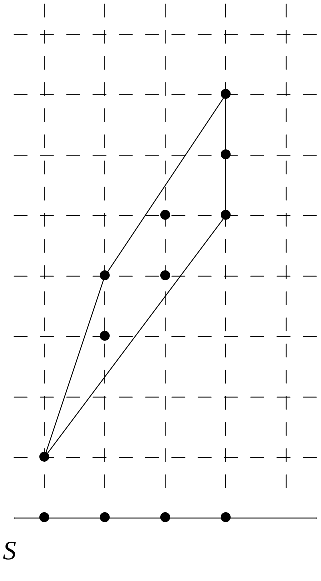


Figure 5.80: A transformed polytope and its integer projection

be “small”. In particular, if the dimension m is fixed, then the lattice width (see subsection 5.22) of lattice point free polytopes is bounded by a constant $\omega(m)$ (Lagarias et al. 1990; Barvinok 2002; Banaszczyk et al. 1999). This means that in the direction of the lattice width of a polytope, the gaps will be not be larger than $\omega(m)$ (Barvinok and Woods 2003, Theorem 4.3). Otherwise, we would be able to put a (uniformly) scaled down version of the polytope in the gap and it would contain no lattice points, which would contradict the fact that its lattice width is bounded by $\omega(m)$. Figure 5.78 contains such a scaled down copy of the original polytope. However, neither the horizontal nor the vertical direction is a lattice width direction of this polytope. The actual lattice width of this polytope was computed in Example 5.68 as 3 with corresponding direction $\mathbf{c} = (-1, 2)$. Figure 5.80 shows the result of applying the unimodular transformation

$$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$$

to the polytope. Note that the horizontal direction now has gaps of width at most 1. After shifting, subtracting and projecting in the vertical direction, we therefore end up with a set S with gaps of width 1 and we then only have to shift and subtract once in the remaining (horizontal) direction.

In fact, for two-dimensional polytopes the gaps in the lattice width direction will always be one, as shown by the following lemma.

Lemma 5.81 *For any rational polygon, the gaps in a lattice width direction are of width at most 1.*

Proof We may assume that x is the given lattice width direction of a given polygon P. If there is a gap of width 2, then there is an integer value x_1 of x such that $P \cap \{\, (x_1, y) \,\} \neq \emptyset$, $P \cap \{\, (x_1 + 2, y) \,\} \neq \emptyset$, while $P \cap \{\, (x_1 + 1, y) \,\} \cap \mathbb{Z}^2 = \emptyset$. Using Barvinok and Woods (2003, Lemma 4.2), we can put a scaled down copy P' of P between $x = x_1$ and $x = x_1 + 2$ (and inside of P). P' meets the line $x = x_1 + 1$ between two consecutive integer points, y_1 and $y_1 + 1$. Let P'' be the polygon bounded by $x = x_1$ and $x = x_1 + 2$ and two lines that separate P' from these two integer points. P'' will have the same width (2) in the x direction, while $P' \subset P''$. The x direction is therefore also a lattice width direction of P''. P'' cannot intersect both $x = x_1$ and $x = x_1 + 2$ in a segment of length greater than or equal to 1. Otherwise, it would also intersect $x = x_1 + 1$ in a segment of length greater than or equal to 1.

We may therefore assume that the length of the intersection of P'' with $x = x_1$ is smaller than 1. If this line segment contains an integer point, then call it y_2. Otherwise, let y_2 be the greatest integer smaller than the points in the line segment. We may assume that $y_1 = y_2$. Otherwise, we can apply the unimodular transformation

$$\begin{bmatrix} x \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ y_1 - y_2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

without changing the width in direction x. If P'' contains (x_1, y_1), it intersects $x = x_1$ in a segment $[y_1 - \alpha_1, y_1 + \alpha_2]$. We may then similarly assume that $\alpha_2 \geq \alpha_1$. P'' will only cut $x = x_1 + 2$ in points with y-coordinate smaller than $2 - \alpha_2$. The width in the y direction will therefore be smaller than $2 - \alpha_2 + \alpha_1 \leq 2$, contradicting that x is a

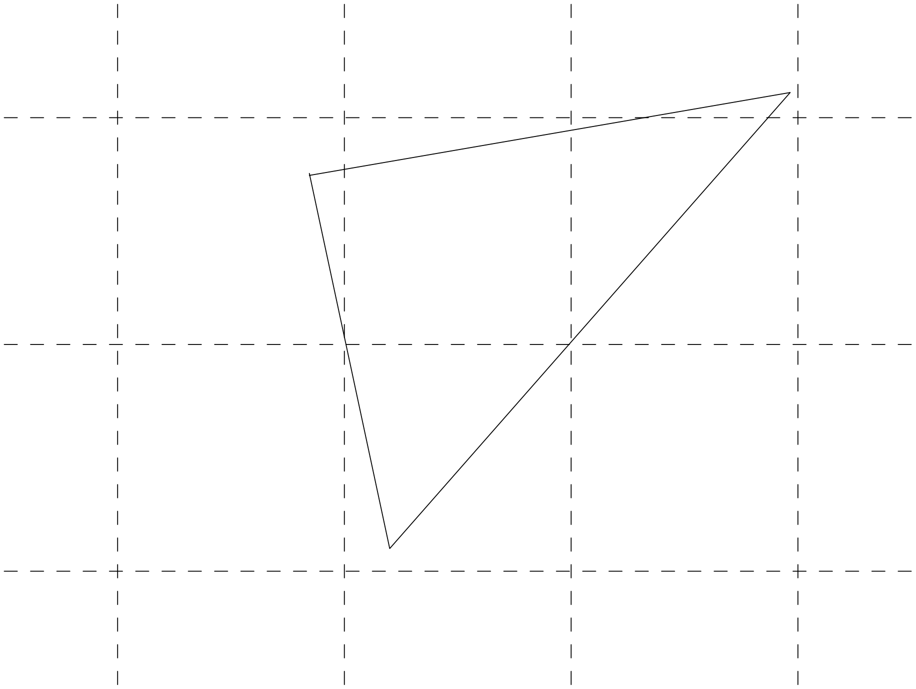


Figure 5.82: Lattice point free polygon with lattice width 2

lattice width direction. If P'' does not contain (x_1, y_1), then it only intersects $x = x_1$ in points with y-coordinate $y_1 + \alpha$ with $0 < \alpha < 1$. Given any such point, it is clear that P'' intersects $x = x_1 + 2$ only in points with y-coordinate strictly between $y_1 - \alpha$ and $y_1 + 1 - \alpha$, again showing that the width in the y direction is smaller than 2 and leading to the same contradiction. The contradiction shows that there can be no gaps of width 2 in the lattice width direction of P. $\square$

Note that the $\omega(2)$ bound is too coarse to reach the above conclusion as $\omega(2) > 2$. An example of a polygon with lattice with greater than 2 is the polygon with vertices $(-17/110, 83/110)$, $(2/10, -9/10)$ and $(177/90, 100/90)$, shown in Figure 5.82, which has width 221/110.

The idea of the projection algorithm is now to first find a direction in which the gaps are expected to be small and to unimodularly transform the existentially quantified variables such that this direction lies in the direction of one of the transformed variables. Then, the remaining existentially quantified variables are projected out by applying the technique recursively. The resulting generating function will have gaps at most $\omega(m)$ wide and so we have to subtract at most $\omega(m)$ shifted copies of this generating function before we can plug in 1 to project out the selected (and now only remaining) existentially quantified variable. We now look at each of these step in a bit more detail.

We are given a polyhedron P such that $P_{\mathbf{s}}$ is a polytope and we want to compute a generating function $f(\mathbf{x}, \mathbf{y})$ for the set T in (5.79). We first compute the lattice width directions of the m-dimensional parametric polytope $P_{\mathbf{s},\mathbf{t}}$ as in subsection 5.22. The result is a partition of the parameter domain, i.e., the projection of P onto the first $n+d$ coordinates, into partially open polyhedra Q_i, together with the lattice width direction $\mathbf{c}_i$ corresponding to each Q_i. Since the generating functions only encode integer points, we can replace each open facet $\langle \mathbf{a}, \mathbf{x} \rangle + b > 0$ by the closed facet $\langle \mathbf{a}, \mathbf{x} \rangle + b - 1 \geq 0$ to obtain a collection of closed polyhedra $\tilde{Q}_i$. Now let

$$P_i = P \cap \tilde{Q}_i \times \mathbb{Q}^m$$

and let $f_i(\mathbf{x}, \mathbf{y})$ be the generating function of the set

$$T_i = \{ (\mathbf{s}, \mathbf{t}) \mid \exists \mathbf{u} \in \mathbb{Z}^m : (\mathbf{s}, \mathbf{t}, \mathbf{u}) \in P_i \}.$$

Then clearly,

$$f(\mathbf{x}, \mathbf{y}) = \sum_i f_i(\mathbf{x}, \mathbf{y}).$$

From now on, we will consider a particular P_i with corresponding lattice width $\mathbf{c}_i$ and drop the i subscript.

We are now given a polyhedron P such that the lattice width direction of $P_{\mathbf{s},\mathbf{t}}$ is $\mathbf{c}$. We first extend $\mathbf{c}$ to an $m \times m$ unimodular matrix U using the technique of subsection 5.7,

$$U = \begin{bmatrix} \mathbf{c}^T \\ U' \end{bmatrix}$$

and then compute

$$P' = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_d & 0 \\ 0 & 0 & U \end{bmatrix} P.$$

We have

$$T = \{ (\mathbf{s}, \mathbf{t}) \mid \exists \mathbf{u}' \in \mathbb{Z}^m : (\mathbf{s}, \mathbf{t}, \mathbf{u}') \in P' \},$$

i.e., we may have changed the values of the existentially quantified variables, but we have not changed the set T. Now consider the set

$$T' = \{ (\mathbf{s}, \mathbf{t}, u'_1) \mid \exists (u'_2, \dots, u'_m) \in \mathbb{Z}^{m-1} : (\mathbf{s}, \mathbf{t}, \mathbf{u}') \in P' \}.$$

This set has only $m-1$ existentially quantified variables, so we may apply this projection algorithm recursively and obtain the generating function $f'(\mathbf{x}, \mathbf{y}, z)$ for T'. The set T' may no longer correspond to the integer points in a polytope, but, by construction, the gaps in the final coordinate are small ($\leq \omega(m)$).

By now we have a generating function $f'(\mathbf{x}, \mathbf{y}, z)$ for the set T' (with small gaps in the final coordinate) and we have to compute the generating function $f(\mathbf{x}, \mathbf{y})$ for T. By computing

$$f''(\mathbf{x}, \mathbf{y}, z) = f'(\mathbf{x}, \mathbf{y}, z) \bigoplus_{k=1}^{\lfloor \omega(m) \rfloor} \left(z^k f'(\mathbf{x}, \mathbf{y}, z) \right), \tag{5.83}$$

where $\oplus$ represents the operation on generating functions that corresponds to set difference on the corresponding sets, we obtain a generating for the set T'' where only the smallest value of u'_1 is retained. The total number of u'_1s associated to any $(\mathbf{s}, \mathbf{t})$ is therefore either zero or one and so the "multiset" defined by taking as many copies of $(\mathbf{s}, \mathbf{t})$ as there are associated values of u'_1 is actually the set T. That is

$$f(\mathbf{x}, \mathbf{y}) = f''(\mathbf{x}, \mathbf{y}, 1).$$

The only remaining problem is that the "$\oplus$" operation in (5.83) is fairly expensive. In particular, this operation is performed by first computing the Hadamard product of the two generating functions (which corresponds to the intersection of the sets) and then subtracting the resulting generating function from this first generating function. The last operation is fairly cheap, but the Hadamard product has a time complexity which while polynomial if the dimension (in this case the maximum of k_i in (5.26)) is fixed, is exponential in this dimension. Furthermore, this dimension increases by $\max k_i - d$ on each successive application of the Hadamard product, while $\max k_i > d$ as soon as some projection is performed, which certainly happens in the recursive application of the algorithm. Now, the total number of Hadamard products is bounded by a constant $\lfloor \omega(m) \rfloor$ and so the increase in dimension is also bounded by a constant, so the whole operation is still polynomial for fixed dimension. Nevertheless, we do not want to perform any additional Hadamard products if we do not really have to. That is, we would like to be able to detect when we can stop performing these operations *before* reaching the upper bound $\lfloor \omega(m) \rfloor$.

Let $f'_0(\mathbf{x}, \mathbf{y}, z) = f'(\mathbf{x}, \mathbf{y}, z)$ and let $f'_k(\mathbf{x}, \mathbf{y}, z)$ be the result of applying the "set difference" in (5.83) k times. Denote the corresponding sets by T'_0 and T'_k. We want to find the smallest k such that $f''(\mathbf{x}, \mathbf{y}, z) = f'_k(\mathbf{x}, \mathbf{y}, z)$. Note that we are talking about equality of rational functions here. Any further application of the set difference operation will not change this rational function, but it *will* typically produce a different (more complex) representation. To check whether the current k is sufficient, we are going to count how many times any element of T'_k still appears in a shifted copy of T'_0, with shift greater than or equal to $k + 1$. If this number is zero, any further set difference will have no effect. That is, we want to compute

$$\sum_{l=k+1}^{\infty} \left| T'_l \cap (\mathbf{e}_{n+d+1} + T') \right|,$$

or, in terms of generating functions,

$$h(\mathbf{x}, \mathbf{y}, z) = \sum_{l=k+1}^{\infty} f'_k(\mathbf{x}, \mathbf{y}, z) \star z^l \, f'(\mathbf{x}, \mathbf{y}, z).$$

We should point out here that while the Hadamard product ($\star$) corresponds to intersection when applied to generator functions of indicator functions (i.e., with coefficients one or zero), in general it will produce a generating function with coefficients that are the product of the corresponding coefficients in the two operands. We can therefore rewrite the above equation as

$$\begin{aligned}
h(\mathbf{x}, \mathbf{y}, z) &= \sum_{l=k+1}^{\infty} f'_k(\mathbf{x}, \mathbf{y}, z) \star z^l\, f'(\mathbf{x}, \mathbf{y}, z) \\
&= f'_k(\mathbf{x}, \mathbf{y}, z) \star \left(\sum_{l=k+1}^{\infty} z^l\, f'(\mathbf{x}, \mathbf{y}, z) \right) \\
&= f'_k(\mathbf{x}, \mathbf{y}, z) \star \frac{z^{k+1}\, f'(\mathbf{x}, \mathbf{y}, z)}{1 - z}.
\end{aligned}$$

Computing $h(\mathbf{x}, \mathbf{y}, 1)$ would give us a generating function with as coefficients how many times a point of T'_k still appears in a shifted copy of T'_0 for any particular value of $\mathbf{s}$ and $\mathbf{t}$. However, we want to know if this number is zero for *all* values of $\mathbf{s}$ and $\mathbf{t}$, so we compute $h(\mathbf{1}, \mathbf{1}, 1)$ instead. We have to be careful here since we allow the polyhedron P to be unbounded and so we should apply the technique of subsection 5.23 with Q the projection of P on the remaining coordinates.

Note that testing whether we can stop is more expensive than applying the next iteration (since we have an extra $(1 - z)$ factor in the denominator of one of the operands). However, we may save many iterations by stopping early and we will not needlessly replace a given representation of $f''(\mathbf{x}, \mathbf{y}, z)$ by a more complex representation (with more factors in the denominator). An alternative way of checking whether we can stop is to test whether $f'_k(\mathbf{x}, \mathbf{y}, z) = f'_{k+1}(\mathbf{x}, \mathbf{y}, z)$ (as rational functions). To do so, we would need to check that both

$$f'_k(\mathbf{x}, \mathbf{y}, z) - \left(f'_k(\mathbf{x}, \mathbf{y}, z) \star f'_{k+1}(\mathbf{x}, \mathbf{y}, z) \right)$$

and

$$f'_{k+1}(\mathbf{x}, \mathbf{y}, z) - \left(f'_k(\mathbf{x}, \mathbf{y}, z) \star f'_{k+1}(\mathbf{x}, \mathbf{y}, z) \right)$$

are zero and this Hadamard product is more expensive than the one above.

Example 5.85 *Consider once more the parametric polytope*

$$P(p) = \left\{ \mathbf{x} \mid \begin{array}{r} -2x_1 + p + 5 \geq 0 \\ 2x_1 + p + 5 \geq 0 \\ -2x_2 - p + 5 \geq 0 \\ 2x_2 - p + 5 \geq 0 \end{array} \right\}$$

from Woods (2004, Example 2.1.7) and Example 5.73 and assume we want to compute

$$c(p) = \left[\exists \mathbf{x} \in \mathbb{Z}^2 : (p, \mathbf{x}) \in P \right].$$

That is, we simply want to know for which values of p the polytope is non-empty. Now, there are more efficient ways of answering this particular question, but we will use it here as an example of the above algorithm. The polytope $P(p)$ is shown in Figure 5.84 for all integer value of the parameter for which the polytope is non-empty.

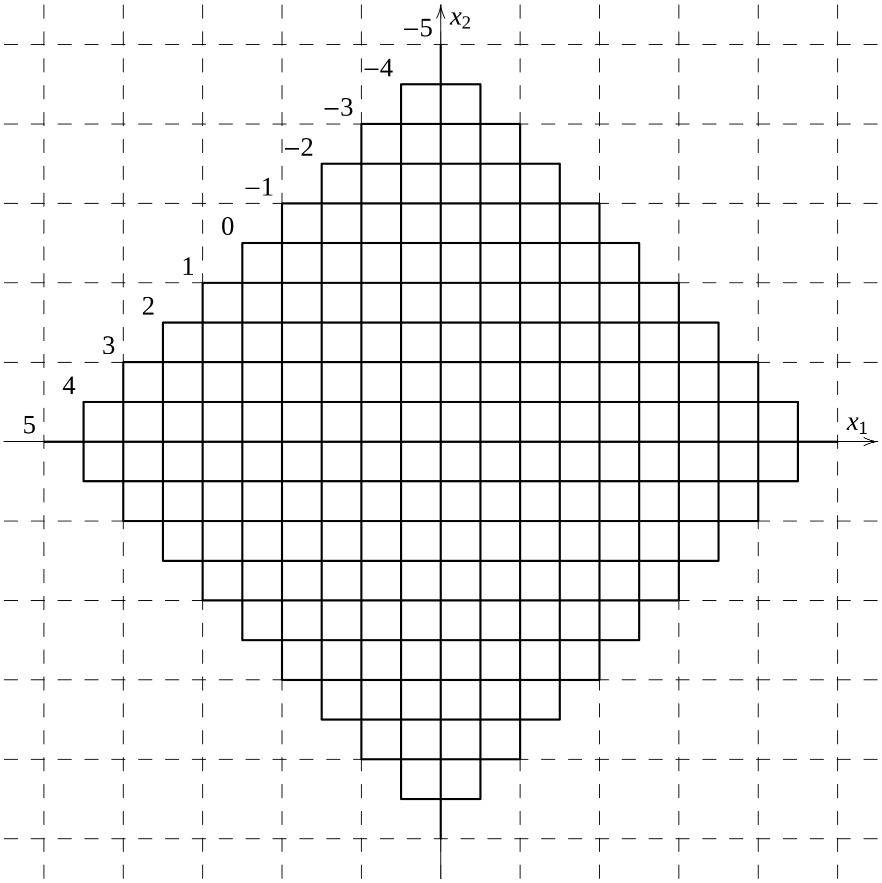


Figure 5.84: The parametric polytope from Example 5.85 for different values of the parameter

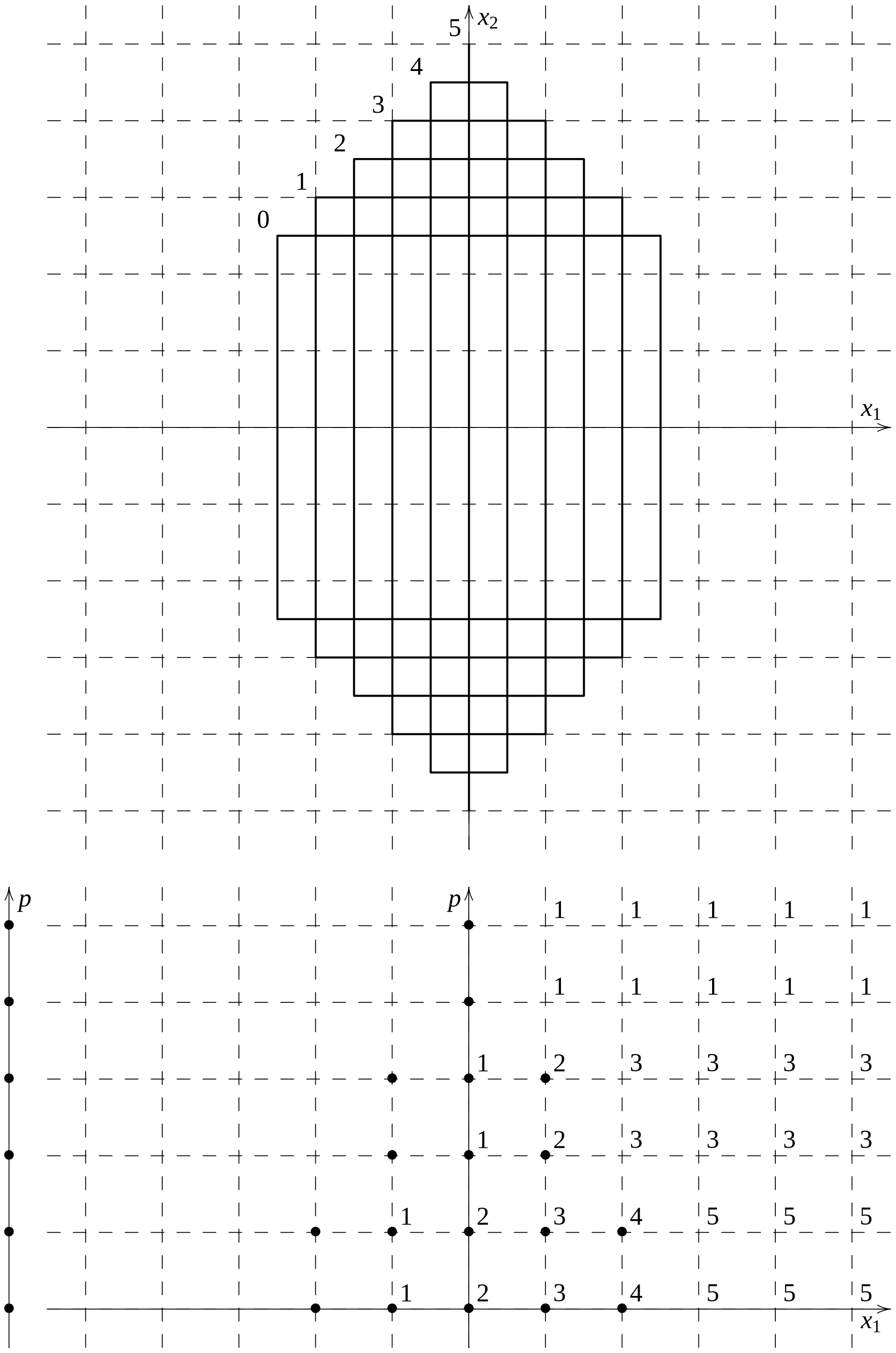


Figure 5.86: The transformed parametric polytope from Example 5.85 for $0 \leq p \leq 5$

The first step is to compute the lattice width of $P(p)$. In Example 5.73, we already obtained the decomposition of the parameter domain into

$$\begin{aligned}\hat{C}_1 &= \{\, p \mid 0 \le p \le 5 \,\}\\ \hat{C}_2 &= \{\, p \mid -5 \le p \le -1 \,\},\end{aligned}$$

with corresponding lattice widths and lattice width directions

$$\begin{aligned}\mathbf{c}_1 &= (0,1) \quad w_{\mathbf{c}_1}(p) = 5 - p\\ \mathbf{c}_2 &= (1,0) \quad w_{\mathbf{c}_2}(p) = 5 + p.\end{aligned}$$

Note that in this example, the gaps in both coordinate directions are 1*, so, in principle, there is no need to perform any unimodular transformation here. Nevertheless, we will apply the transformation that would be applied by the algorithm. On the first domain, we extend the lattice width direction to the unimodular matrix*

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

After application to the existentially quantified variables $\mathbf{x}$*, we obtain the parametric polytope*

$$P'_1(p) = \left\{ \mathbf{x} \mid \begin{aligned} -2x_2 + p + 5 &\ge 0\\ 2x_2 + p + 5 &\ge 0\\ -2x_1 - p + 5 &\ge 0\\ 2x_1 - p + 5 &\ge 0\\ p &\ge 0 \end{aligned} \right\}$$

shown in the top part of Figure 5.86. We now temporarily remove the existential quantification on x_1*, resulting in*

$$T' = \{(p, x_1) \in \mathbb{Z}^2 \mid \exists x_2 \in \mathbb{Z} : (s, \mathbf{x}) \in P'\}.$$

Since there is only one existentially quantified variable left, we know we only have to shift and subtract the set once to obtain a set with at most one value of x_2 *associated to each value of* (p, x_1)*. In particular, let* $f(x, z_1, z_2)$ *be the generating function of the integer points in* P'*. Then* $g(x, z_1) = f'(x, z_1, 1)$*, with* $f'(x, z_1, z_2) = f(x, z_1, z_2) - f(x, z_1, z_2) \star z_2 f(x, z_1, z_2)$*, is the generating function of* T'*.*

To check whether we need to subtract any shifted copies of $g(x, z_1)$ *from itself, we compute*

$$h(x, z_1) = g(x, z_1) \star \frac{z_1\, g(x, z_1)}{1 - z_1}.$$

The second argument of this Hadamard product is depicted in Figure 5.86 by its coefficients. The exponents in $h(x, z_1)$ *that have a non-zero coefficient are therefore those marked by both a dot (•) and a number. The total sum can be computed as* $h(1, 1) = 26$*, which is finite, but non-zero. We therefore need to subtract at least one shifted copy of* $g(x, z_1)$*. Let*

$$g'(x, z_1) = g(x, z_1) - g(x, z_1) \star z_1 g(x, z_1).$$

Computing

$$h'(x, z_1) = g'(x, z_1) \star \frac{z_1^2\, g(x, z_1)}{1 - z_1},$$

we would find that $h'(1, 1) = 0$ *and so we do not need to shift and subtract any further. However, since we are dealing with a two-dimensional problem, we already know from Theorem 5.81 that we can stop after one shift and subtract, so we do not even need to compute* $h'(x, z_1)$ *here. The function* $g'(x, z_1)$ *now only has non-zero coefficients for at most one exponent of* z_1 *for each exponent of* x*. We may therefore project down to obtain the function* $g'(x, 1)$*, which is the generating function of the set in the lower left part of Figure 5.86.*

For the chamber $\hat{C}_2$ *of the parameter domain, the computations are nearly identical and the final result is simply the sum of the two generating functions found for each of the two (disjoint) chambers.*

6 Publications

6.1 Publications about the Library

This is a list of some reports and publications explaining details of parts of the `barvinok` library.

- Analytical computation of Ehrhart polynomials and its applications for embedded systems (Verdoolaege, Beyls, Bruynooghe, Seghir, and Loechner; 2004b)
- Analytical computation of Ehrhart polynomials and its applications for embedded systems (Verdoolaege, Beyls, Bruynooghe, Seghir, and Loechner; 2004c)
- Analytical Computation of Ehrhart Polynomials and its Application in Compile–Time Generated Cache Hints (Seghir, Verdoolaege, Beyls, and Loechner; 2004)
- Analytical computation of Ehrhart polynomials: Enabling more compiler analyses and optimizations (Verdoolaege, Seghir, Beyls, Loechner, and Bruynooghe; 2004d)
- Experiences with enumeration of integer projections of parametric polytopes (Verdoolaege, Beyls, Bruynooghe, and Catthoor; 2004a)
- Experiences with enumeration of integer projections of parametric polytopes (Verdoolaege, Beyls, Bruynooghe, and Catthoor; 2005a)
- Computation and Manipulation of Enumerators of Integer Projections of Parametric Polytopes (Verdoolaege, Woods, Bruynooghe, and Cools; 2005b)
- Incremental Loop Transformations and Enumeration of Parametric Sets (Verdoolaege; 2005)
- Symbolic Polynomial Maximization over Convex Sets and its Application to Memory Requirement Estimation (Clauss, Fernández, Gabervetsky, and Verdoolaege; 2006)
- Counting with rational generating functions (Verdoolaege and Woods; 2008)
- Counting integer points in parametric polytopes using Barvinok's rational functions (Verdoolaege, Seghir, Beyls, Loechner, and Bruynooghe; 2007b)
- Polynomial Approximations in the Polytope Model: Bringing the Power of Quasi-Polynomials to the Masses (Meister and Verdoolaege; 2008)
- Bounds on Quasi-Polynomials for Static Program Analysis (Devos, Verdoolaege, Van Campenhout, and Stroobandt; 2007)
- Computing parametric rational generating functions with a primal Barvinok algorithm (Köppe and Verdoolaege; 2008)
- An Implementation of the Barvinok–Woods Integer Projection Algorithm (Köppe, Verdoolaege, and Woods; 2008)

- Algorithms for Weighted Counting over Parametric Polytopes: A Survey and a Practical Comparison (Verdoolaege and Bruynooghe; 2008)

6.2 Publications Refering to the Library

This is a list of some reports and publications refering to the `barvinok` library.

- Theorems of Brion, Lawrence, and Varchenko on rational generating functions for cones (Beck, Haase, and Sottile; 2005)
- Generating Cache Hints for Improved Program Efficiency (Beyls and D'Hollander; 2005)
- An alternative algorithm for counting lattice points in a convex polytope (Lasserre and Zeron; 2005)
- Volume Calculation and Estimation of Parameterized Integer Polytopes (Rabl; 2006)
- Improved Derivation of Process Networks (Verdoolaege, Nikolov, and Stefanov; 2006)
- Computing the Ehrhart quasi-polynomial of a rational simplex (Barvinok; 2006)
- Memory Optimization by Counting Points in Integer Transformations of Parametric Polytopes (Seghir and Loechner; 2006)
- GRAPHITE: Polyhedral Analyses and Optimizations for GCC (Pop, Silber, Cohen, Bastoul, Girbal, and Vasilache; 2006)
- Volume Computation for Polytopes and Partition Functions for Classical Root Systems. (Baldoni-Silva, Beck, Cochet, and Vergne; 2006)
- A primal Barvinok algorithm based on irrational decompositions (Köppe; 2007)
- pn: A Tool for Improved Derivation of Process Networks (Verdoolaege, Nikolov, and Stefanov; 2007a)
- On Ehrhart Polynomials and Probability Calculations in Voting Theory (Lepelley, Louichi, and Smaoui; 2008)
- Local Euler-Maclaurin formula for polytopes (Berline and Vergne; 2006)

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List of Acronyms

GCD	greatest common divisor
HNF	Hermite Normal Form
LCM	least common multiple
LLL	Lenstra, Lenstra and Lovasz' basis reduction algorithm
PIP	Parametric Integer Programming
SNF	Smith Normal Form

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